SUPPLEMENTARY NOTES DR. K. CRAIG

SIMPLE PHYSICAL SYSTEMS

1. MECHANICAL SYSTEMS: Translational and Rotational

(a) Variables

Across Variables: displacements x (L) and θ
velocities v (L/t) and ω (1/t)
accelerations a (L/t²) and α (1/t²)

Through Variables: force f (F)
moment M (F·L)

Inertial Frame of Reference (Absolute) - not accelerating

Degrees of Freedom - number of degrees of freedom of a mechanical system is the minimum number of geometrically independent coordinates required to describe its configuration completely.

Other variables: w energy (F·L)
$p = \dot{w}$ power (F·L/t)

(b) Element Laws

- mass and moment of inertia
- stiffness
- friction
- lever and gears

mass: $m\dot{v} = f$

moment of inertia: $J\dot{\omega} = M$ (rotation about fixed axis)

friction: algebraic relationship between force (torque) and relative (angular) velocity.

VISCOUS: $f = B\Delta v$ $M = B\Delta\omega$

stiffness: algebraic relationship between force (torque) and relative (angular) displacement.

$f = K\Delta x$ $M = K\Delta\theta$

<u>Ideal lever:</u> rigid, no mass, no friction, no stored energy.

<u>Ideal gear:</u> rigid, no moment of inertia, no friction, no stored energy, perfect meshing of teeth.

Ideal lever (gear) transmits energy from one part of the mechanical system to another such that force (torque) level and motion level are altered, but power level is not.

(C) System Relations

(1) Newton's Law and D'Alembert's Principle

General Eqs. of Motion for a rigid body in plane motion:

$$\Sigma \vec{F} = m\vec{\bar{a}} \qquad \Sigma \bar{M} = \bar{I}\alpha$$

Rigid Body Translation ($\alpha = 0$): $\Sigma \vec{F} = m\vec{\bar{a}} \qquad \Sigma \bar{M} = 0$

Fixed Axis Rotation about point O:

$$\Sigma F_n = m\bar{r}\omega^2 \qquad \Sigma F_t = m\bar{r}\alpha \qquad \Sigma \bar{M} = \bar{I}\alpha$$

$$\Sigma M_0 = I_0\alpha$$

($\bar{r}$ = distance from CG to O)

D'Alembert's Principle (Dynamic Equilibrium):

$$\Sigma \vec{F} - m\vec{\bar{a}} = 0 \qquad \Sigma \bar{M} - \bar{I}\alpha = 0$$

$-m\vec{\bar{a}}$ = inertial force

$-\bar{I}\alpha$ = inertial moment

Therefore

$$\Sigma M = 0$$
$$\Sigma \vec{F} = 0$$

(2) Law of Reaction Forces and Torques

Accompanying any force of one element on another, there will be a reaction force on the first element of equal magnitude and opposite direction. For bodies that are rotating about <u>same</u> axis, any torque exerted by one element on another is accompanied by a reaction torque of equal magnitude and opposite direction on first element.

(3) Law for Displacements

In a system of connected mechanical elements we may inspect, along an imaginary closed line through all the connections and back to the starting point, the state of motion of each element, and, because of the connectedness, the individual motions must add up to zero. The motion of the parts must be compatible with the motion of the whole.

$$\left.\begin{array}{l}\sum_i \Delta x_i = 0 \\ \sum_i \Delta \theta_i = 0\end{array}\right\} \text{ around any closed path}$$

(4) Energy Method

1st Law of Thermodynamics: Conservation of Energy

$\Delta E_{sys} = Q_{IN} + W_{IN}$ $\quad$ Q_{IN} = net heat transferred into system

W_{IN} = net work done on system

Purely Mechanical System: no flow of heat through boundaries of system and no energy stored by system in form of heat during process under consideration. Energy within system is in form of either kinetic or potential energy. Energy enters and leaves system only in form of mechanical work. 1st Law becomes:

$$\Delta(T + U) = W_{IN}$$

T = Kinetic Energy of System = $\frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$ for a rigid body in plane motion

U = potential energy of System $\quad$ $U = mg\Delta h$

$U = \frac{1}{2}k(\Delta x)^2$

W_{IN} = net work done on system by all external forces.

Energy Dissipated through Friction: In all real systems there is friction which dissipates energy. Account for this by computing the work done by friction and assuming that the conversion of work to heat takes place outside system. Friction work may not be reconverted to mechanical energy and must be counted as energy lost from system.

(d) State Variables

(1) State Variables - these variables completely describe the effect of the past history of the system on its response in the future. They are independent, i.e., it must be impossible to express any state variable as an algebraic function of the remaining state variables and the inputs. Knowing the values of the state variables at a reference time t_0 and the values of the inputs for all $t \geq t_0$ is sufficient for evaluating the state variables and outputs for all $t \geq t_0$. Choice of state variables is not unique. They are usually related to the energy stored in each of the system's energy storing elements. Since any energy initially stored in these elements can affect the response of the system at a later time, one state variable will normally be associated with each of the energy-storing elements. Sometimes the number of state variables is different from the number of energy-storing elements because a particular interconnection of elements causes redundant variables or because there is a need for a state variable not related to the storage of energy.

(2) State Variables for Mechanical Systems -

Energy stored in mechanical systems must be associated with masses (Kinetic energy $\frac{1}{2}mv^2$ or $\frac{1}{2}I\omega^2$) and springs (potential energy $\frac{1}{2}K(\Delta x)^2$).

State Variables: velocities of masses
elongations of springs

(3) State Variable Equations - Coupled set of 1st order ODE's where the derivative of each state variable is expressed as an algebraic function of state variables,

inputs, and possibly time.

(e) Other Results

(1) Springs in Parallel $\quad K_{eq} = K_1 + K_2$

Springs in Series $\quad K_{eq} = \dfrac{1}{\frac{1}{K_1} + \frac{1}{K_2}}$

(2) Dampers in Parallel $\quad B_{eq} = B_1 + B_2$

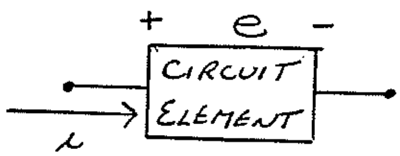
Dampers in Series $\quad B_{eq} = \dfrac{1}{\frac{1}{B_1} + \frac{1}{B_2}}$

2. ELECTRICAL SYSTEMS

(a) Variables:

Through Variable: i current

Across Variable: e voltage



Positive senses of voltage and current for a circuit element.

Power supplied to element $= ei$

If power is negative, then circuit element is supplying power to rest of circuit.

power $p = ei$ (watts)

Energy supplied to element over interval t_0 to t_1:

$$\int_{t_0}^{t_1} p(t)\,dt \quad \text{(unit is joules)}$$

(1 joule = 1 volt-ampere-sec)

(b) Element Laws

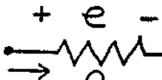
Resistors, Capacitors, Inductors $\}$ passive elements

Sources $\}$ active elements

__passive elements__ - can store or dissipate energy but cannot introduce additional energy.

__active elements__ - can introduce energy into circuit and serve as inputs.

__resistor__ - element for which there is an algebraic relationship between voltage across its terminals and current through it.



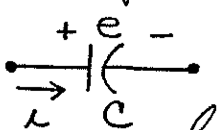
linear resistor: $e = Ri$ (Ohm's Law)

R = Resistance (Ohms)

Power dissipated by linear resistor $= ei$

$$= Ri^2 = \frac{1}{R}e^2$$

__capacitor__ - element that obeys an algebraic relationship between voltage and charge.



linear capacitor $q = Ce$ or $i = C\frac{de}{dt}$

C = Capacitance (farads)

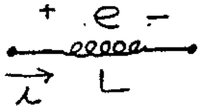
$$e(t) = e(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\lambda)\,d\lambda$$

Energy supplied to a capacitor is stored in its electrical field and can affect the response of a circuit at future times.

Energy stored in a fixed linear capacitor: $\frac{1}{2}Ce^2$

Because energy stored is a function of voltage across its terminals, the initial voltage $e(t_0)$ of a capacitor is one of the conditions we need to find complete response of a circuit for $t \geq t_0$.

__inductor__ - element for which there is an algebraic relationship between the voltage across its terminals and the derivative of the flux linkage.



linear inductor: $e = L \frac{di}{dt}$

L = inductance (henries)

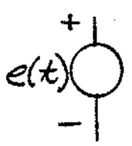
$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} e(\lambda)\,d\lambda$$

Energy supplied to an inductor is stored in its magnetic field.

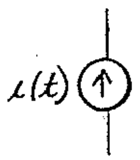
Energy stored in an inductor: $\frac{1}{2} L i^2$

To find the complete response of a circuit for $t \geq t_0$, we need to know the initial current $i(t_0)$ for each inductor.

Sources



voltage source - any device that causes a specified voltage to exist between 2 points in a circuit, regardless of current that may flow.



current source - causes a specified current to flow through branch containing the source, regardless of the voltage that may be required.

We often represent physical sources by the combination of an ideal source and a resistor.

Open and Short Circuits

open circuit - any element through which current cannot flow, e.g. switch in an open position.

short circuit - any element across which there is no voltgage, e.g. switch in a closed position.

(c) System Relations

(1) Kirchhoff's Voltage Law (KVL): If a closed path (loop) is traced through any part of a circuit, the algebraic sum of the voltages across the elements comprising the loop must equal zero.

(2) Kirchhoff's Current Law (KCL): The algebraic sum of the currents at any node must be zero at all times.

Elements in Series: same current flowing through them.

Elements in Parallel: same voltage across each element.

Two General Procedures for obtaining differential equations:

(1) Loop Eq. Method - Express current through every element in terms of one or more loop currents. Write a set of simultaneous equations by using KVL and element laws.

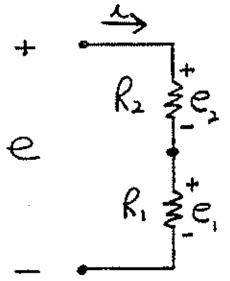
(2) Node Eq. Method - Express voltage across every element in term of node voltages. Write a set of simultaneous eqs. using KCL and element laws.

(d) Important Results

(1) In Steady State inductor behaves as a short circuit since $e = L\frac{di}{dt} = 0$.

(2) In steady state capacitor behaves as an open circuit since $i = C\frac{de}{dt} = 0$.

(3) Resistors in Series

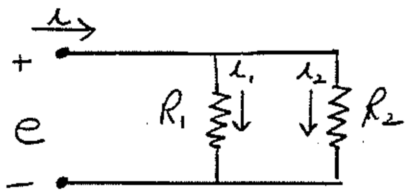


$$R_{eq} = R_1 + R_2 \qquad e = e_1 + e_2$$

$$\left.\begin{aligned} e_1 &= \left(\frac{R_1}{R_1+R_2}\right)e \\ e_2 &= \left(\frac{R_2}{R_1+R_2}\right)e \end{aligned}\right\} \text{Voltage Divider Rule}$$

$$\frac{e_1}{e_2} = \frac{R_1}{R_2}$$

(4) Resistors in Parallel



$$i = i_1 + i_2 \qquad R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\left.\begin{aligned} i_1 &= \left(\frac{R_2}{R_1+R_2}\right)i \\ i_2 &= \left(\frac{R_1}{R_1+R_2}\right)i \end{aligned}\right\} \text{Current Divider Rule}$$

$$\frac{i_1}{i_2} = \frac{R_2}{R_1}$$

(5) Energy stored in an inductor or capacitor cannot change instantaneously unless there is an infinitely large power flow. Therefore neither the voltage across a capacitor nor the current through an inductor can change instantaneously. Capacitor voltages and inductor currents must be continuous functions at all times, as long as all voltages and currents are finite. Other variables can change instantaneously and thus can have discontinuities, e.g. resistor voltages and currents, capacitor currents, inductor voltages.

(6) State Variables – Capacitor voltages and inductor currents are generally selected as state variables.

3. ELECTROMECHANICAL SYSTEMS

Electrical - Mechanical Coupling Considered:

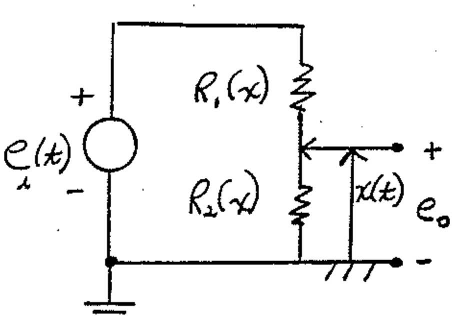
(a) mechanically varying a resistance.

(b) moving a current-carrying conductor in a magnetic field.

(c) varying a capacitance that has an electrical field between plates.

(a) Resistive Coupling

We can control a variable resistance by mechanical motion either continuously by moving an electrical contact or discretely by opening and closing a switch. Resistors cannot store energy, and therefore this method of coupling electrical and mechanical parts of a system does NOT involve mechanical forces ~~that~~ depend on electrical variables.



Equivalent Circuit of a Potentiometer
Resistances R_1 and R_2 depend on wiper position.
Assume NO current flows through the wiper.

$$e_o = \left(\frac{R_2}{R_1+R_2}\right) e_i(t)$$

$$R_2 = \left(\frac{R_1+R_2}{x_{MAX}}\right) x(t)$$

$$e_o = \left(\frac{x(t)}{x_{MAX}}\right) e_i(t)$$

$$0 \leq \frac{x(t)}{x_{MAX}} \leq 1$$

(b) Coupling by a Magnetic Field

Physical laws governing electromechanical coupling where current-carrying wires move within a magnetic field are:

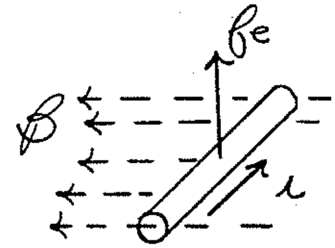
(1) wire in a magnetic field that carries a current will have a force exerted on it.

(2) a voltage will be induced in a wire that moves relative to the magnetic field.

Variables required:

- f_e force on conductor (N)
- v velocity of conductor (m/s)
- ϕ flux (Wb)
- B flux density of magnetic field (Wb/m²)
- i current in conductor (A)
- e_m voltage induced in conductor (V)

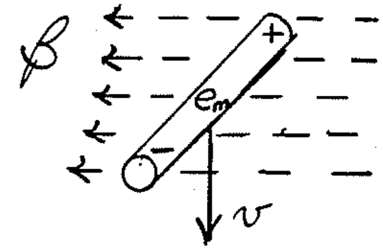
(1) Force on a conductor of differential length dl carrying a current i in a magnetic field of flux density B is



$$\vec{df_e} = i\,(\vec{dl} \times \vec{B})$$

To obtain the total electrically induced force f_e, we integrate along length of the conductor.

(2) Voltage induced in a conductor of differential length dl moving with velocity v in a field of flux density B is $de_m = (\vec{v} \times \vec{B}) \cdot \vec{dl}$



We obtain the total induced voltage for a conductor by integrating between the ends of the conductor.

For a wire moving $\perp$ a magnetic field, these equations reduce to:

$$f_e = \beta l i$$

$$e_m = \beta l v$$

Consider the <u>power involved in coupling mechanism</u>.
External power delivered to electrical part is:

$$P_e = e_m i = (\beta l v) i$$

Power available to whatever mechanical elements are attached to the coil is:

$$P_m = f_e v = (\beta l i) v$$

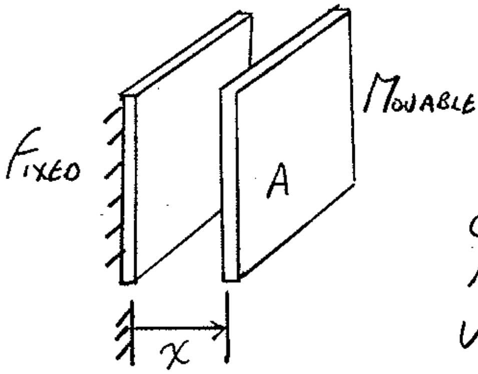
Hence $P_m = P_e$.
Power delivered to coupling mechanism in electrical form will pass on undiminished to mechanical portion. Of course, any practical system has losses resulting from resistance of conductor and friction between moving mechanical elements. However any dissipative elements can be modeled separately by a resistor in the electrical circuit or a viscous friction element acting on a mass.

<u>Examples of Devices</u> coupled by Magnetic Fields:

Galvanometer
Microphone
DC Motor

(C) Coupling by an Electrical Field

Capacitance of two parallel plates of area A, separated a distance d by a dielectric material with permittivity ε, is $C = \frac{\varepsilon A}{d}$, providing that fringing effects are neglected. We can achieve electro-mechanical coupling by making either the distance d or the effective area A vary with time by mechanical means.



$$C = \frac{\varepsilon A}{x}$$

Provided that the permittivity of the dielectric material between the plates is constant (i.e. not affected by the voltage between the plates)

$q = C(x)\, e$ $\qquad$ q = charge on each plate

e = voltage between plates

Energy is stored in the electrical field that exists between the plates of a charged capacitor, and a force is exerted on the plates by an electrical field. Thus any variation in x will involve a force acting on the movable capacitor plate, and there will be a flow of power between the mechanical and electrical parts. For a linear capacitor, the force due to the electrical field is $f_e = \frac{1}{2} e^2 \frac{dC}{dx}$

where the positive sense of f_e is the same as that of x.

$$C = \frac{\varepsilon A}{x}$$

$$\frac{dC}{dx} = -\frac{\varepsilon A}{x^2}$$

$$f_e = -\frac{1}{2}\varepsilon A \frac{e^2}{x^2} \quad \left\{ \begin{array}{l} \text{electrically} \\ \text{induced force} \end{array} \right.$$

$$= -\frac{1}{2\varepsilon A} q^2 \quad \left\{ \begin{array}{l} \text{since } C = \frac{\varepsilon A}{x} \\ \text{and } q = Ce \end{array} \right.$$