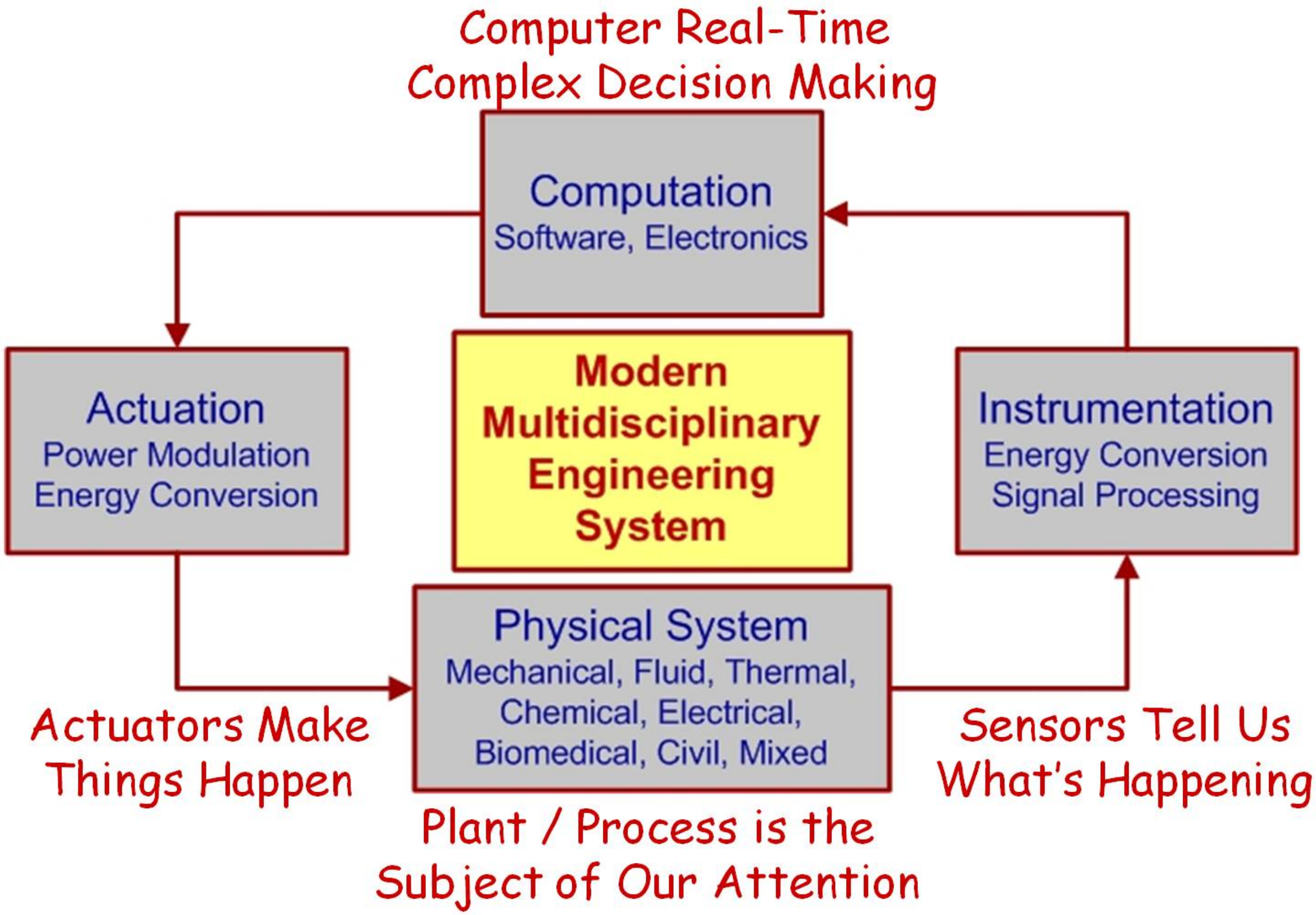
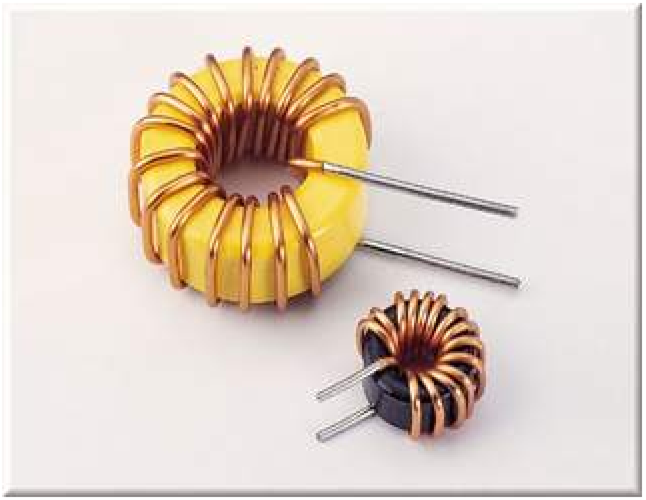
Electrical Systems:
Modeling, Analysis, Measurement, & Control

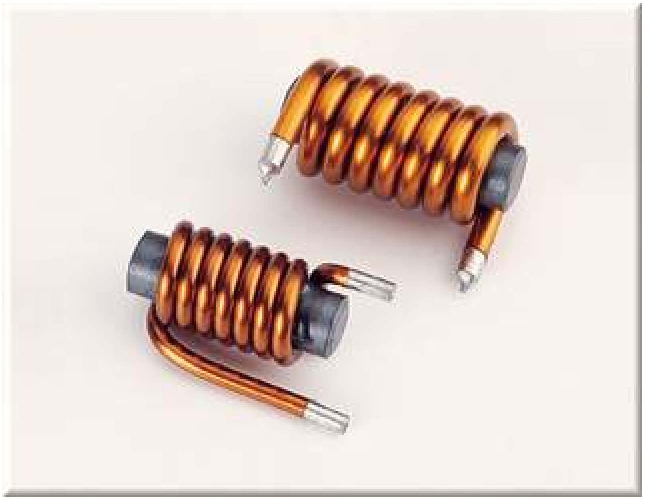


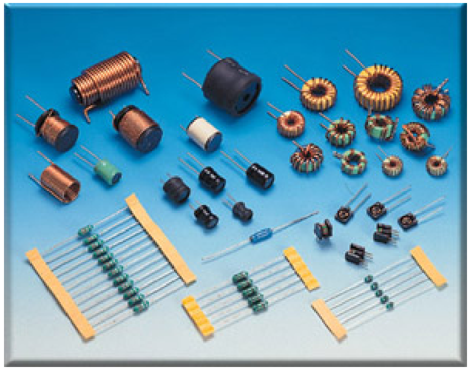
Electrical System Topics

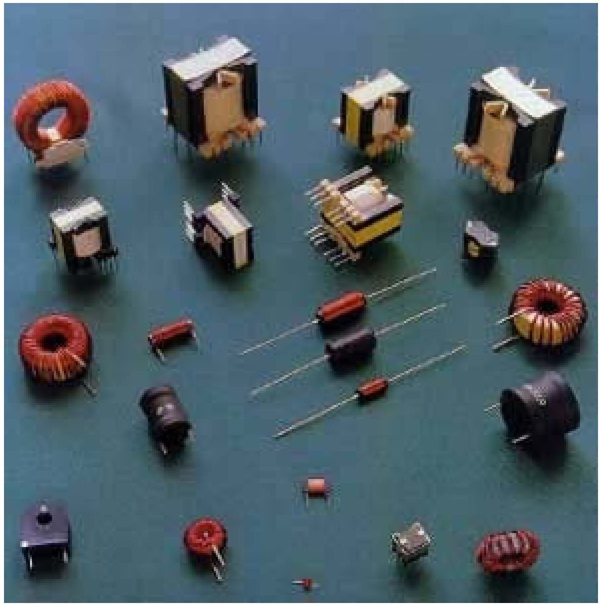
- Part 3
  - Inductor
    - Physical Model
    - Mathematical Model
    - Step Response and Frequency Response
    - Important Uses
  - Electrical Impedance
  - LR and LRC Circuit System Investigations
  - LC Circuit Resonance; Spring / Mass Analogy

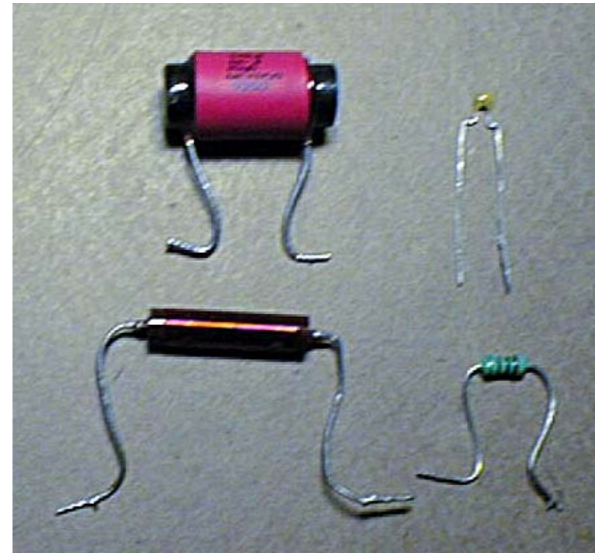


Inductors



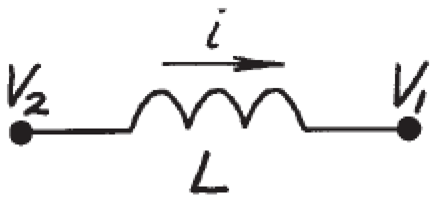






The Inductance Element

- This is another fundamental electrical element. Like a resistor and capacitor, it is intentionally or unintentionally present in every real electrical system.
- An electric current always creates an associated magnetic field (Ampere's Law). If a coil or other circuit lies within this field, and if the field changes with time, an electromotive force (voltage) is induced in the circuit. (Faraday's Law of Induction)

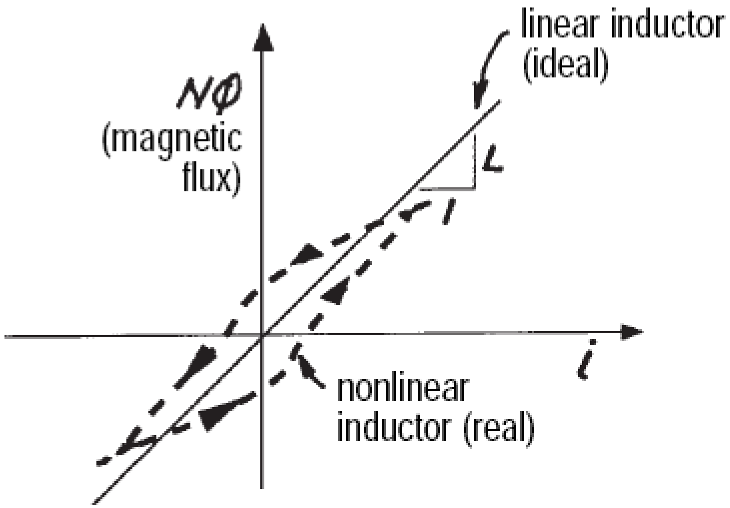


- The magnitude of the induced voltage is proportional to the rate of change of flux $d\phi/dt$ linking the circuit, and its polarity is such as to oppose the cause producing it. (Lenz's Law)
- If no ferromagnetic materials (e.g., iron) are present, the rate of change of flux is proportional to the rate of change of current which is producing the magnetic field.
- The proportionality factor relating the induced emf (voltage) to the rate of change of current is called the *inductance L*.
- The presence of ferromagnetic materials greatly increases the strength of the effects, but also makes them significantly nonlinear, since now the flux produced by the current is not proportional to the current.

- Thus, iron can be used to get a large value of inductance, but the value will be different for different current levels.
- The pure inductance element has induced voltage *e* instantaneously related to *di/dt*, but the relation can be nonlinear.
- The pure and ideal element has *e* directly proportional to *di/dt* (*e* = *L di/dt*), i.e., it is linear and free from resistance and capacitance.

$$e = L\frac{di}{dt}$$

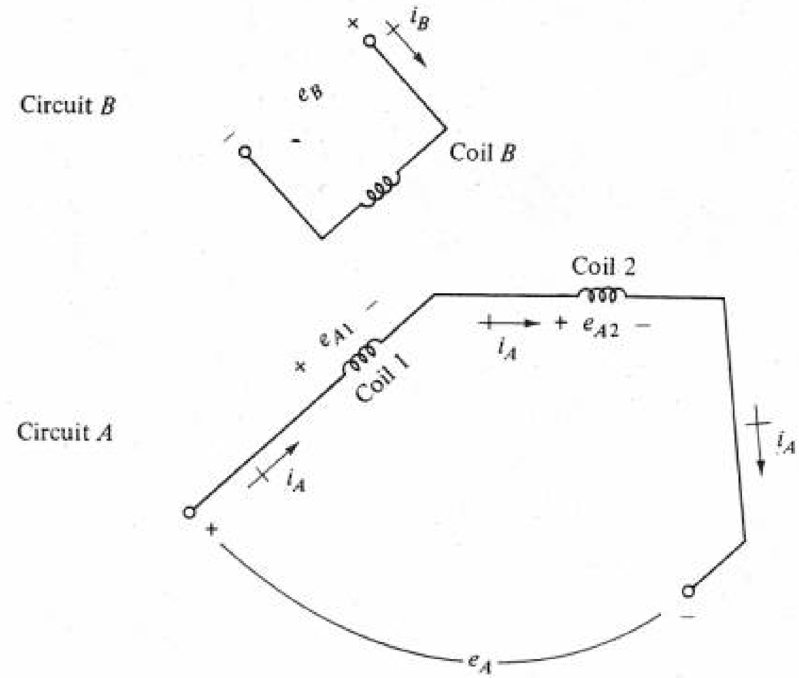
$$L = \frac{e}{\left(\frac{di}{dt}\right)} = \frac{\text{volts}}{\left(\frac{\text{amps}}{\text{sec}}\right)} = \text{henry (H)}$$



- A real inductor always has considerable resistance. At DC and low frequencies, all real inductors behave like resistors, not inductors.
- At high frequencies, all real devices (R, C, L) exhibit complex behavior involving some combination of all three pure elements.
- Thus, real inductors deviate from the pure/ideal model at both low and high frequencies, whereas R and C deviate mainly at high frequencies.
- One can expect real inductors to nearly follow the pure model only for some intermediate range of frequencies and, if the inductance value is small enough to be achieved without the use of magnetic material, the behavior may also approximate the ideal (linear).

- Self-Inductance and Mutual-Inductance
  - Self-inductance is a property of a single coil, due to the fact that the magnetic field set up by the coil current links the coil itself.
  - Mutual inductance causes a changing current in one circuit to induce a voltage in another circuit.
  - Mutual inductance is symmetrical, i.e., a current changing with a certain *di/dt* in coil 1 induces the same voltage in coil 2 as would be induced in coil 1 by the same *di/dt* current change in coil 2. This holds for coils in the same circuit or in separate circuits.
  - The induced voltage in circuit A due to current change in B can either add or subtract from the self-induced voltage in A. This depends on actual geometry.

$$
\begin{aligned}
e_A &= e_{A1} + e_{A2} \\
&= L_1 \frac{di_A}{dt} \pm M_{B/A1} \frac{di_B}{dt} \pm M_{A2/A1} \frac{di_A}{dt} \\
&\quad + L_2 \frac{di_A}{dt} \pm M_{B/A2} \frac{di_B}{dt} \pm M_{A1/A2} \frac{di_A}{dt} \\
&= \left(L_1 + L_2 \pm M_{A2/A1} \pm M_{A1/A2}\right) \frac{di_A}{dt} \\
&\quad + \left(\pm M_{B/A1} \pm M_{B/A2}\right) \frac{di_B}{dt}
\end{aligned}
$$



$M_{A2/A1} = M_{A1/A2} =$ mutual inductance of coils 1 and 2

$L_1 =$ self-inductance of coil 1

$L_2 =$ self-inductance of coil 2

$M_{B/A1} =$ mutual inductance of coils B and A_1

$M_{B/A2} =$ mutual inductance of coils B and A_2

- <u>Energy Stored</u>
  - The pure and ideal inductance stores energy in its magnetic field. The energy stored, irrespective of how the current *i* is achieved, is:

$$\text{Power} = ei = L\frac{di}{dt}i$$

$$\text{Energy} = \int_0^t iL\frac{di}{dt}dt = \int_0^i (Li)\,di = \frac{i^2L}{2}$$

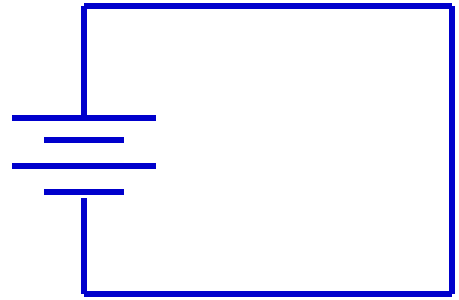
  - If we connect a current-carrying inductor to an energy-using device (e.g., resistor) the inductor will supply energy in an amount ½ Li^2 as its current decays from *i* to 0. During this decay process, *i* if originally positive stays positive, but *di/dt* (and thus *e*) becomes negative, making power negative.

- At very low frequencies, a small voltage amplitude can produce a very large current and thus an inductance is said to approach a short circuit in this case.
- At high frequencies, the current produced by any finite voltage approaches zero, and thus an inductance is said to approach an open circuit at high frequencies.
- For a capacitance, the reverse frequency behavior is observed: the capacitance approaches a short circuit at high frequencies and an open circuit at low frequencies.
- One can often use these simple rules to quickly estimate the behavior of complex circuits at low and high frequency. Just replace L's and C's by open and short circuits, depending on which frequency you are interested in.
- Remember for real circuits that real L's always become R's for low frequency.

Resistors behave the same at all frequencies. Capacitors do not. Inductors do not.

Short Circuit

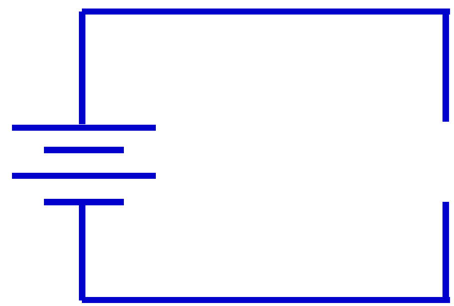
Will carry any current the source can produce until the wire burns up.



Capacitor
at high frequency.
Inductor
at low frequency.

Open Circuit

Will carry no current no matter how large the voltage is, unless arcing occurs.



Capacitor
at low frequency.
Inductor
at high frequency.

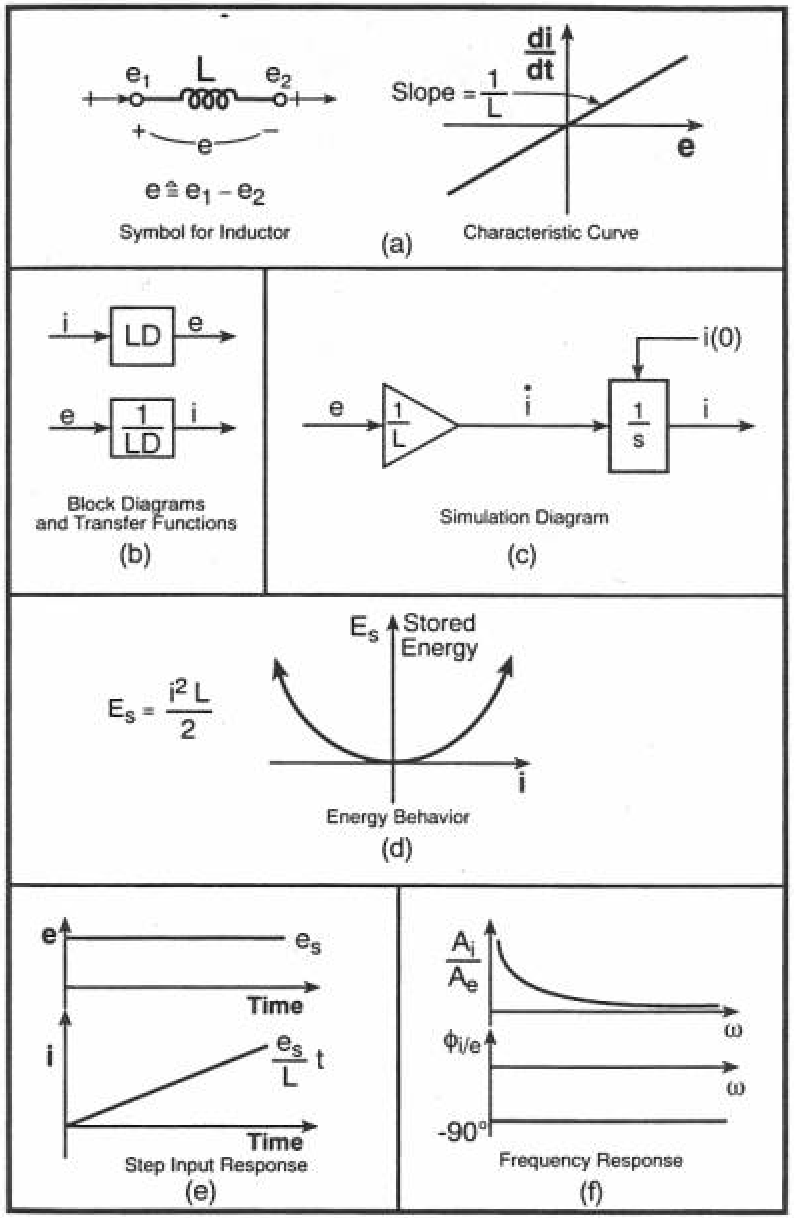
Inductance Element

$$e = L\frac{di}{dt} = LDi$$

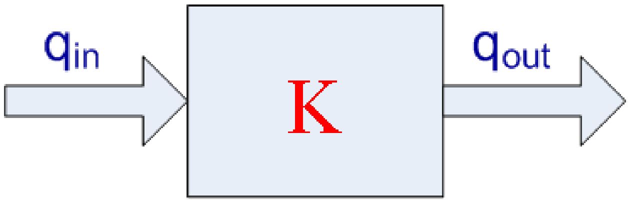
$$\frac{i}{e}(D) = \frac{1}{LD}$$

$$\frac{i}{e}(i\omega) = \frac{1}{i\omega L} = \frac{1}{\omega L}(-i)$$

$$= \frac{1}{\omega L}\angle -90^{\circ}$$

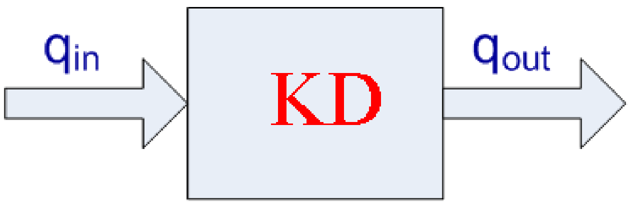


The Three Basic Element Input-Output Relationships



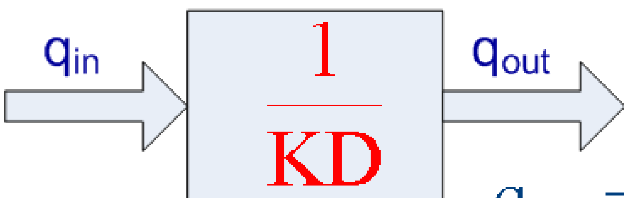
$$q_{out} = Kq_{in}$$

Resistor



$$q_{out} = KDq_{in} = K\frac{dq_{in}}{dt}$$

Inductor



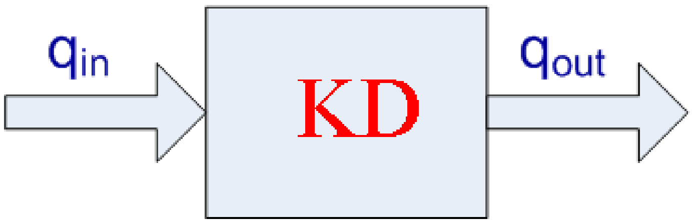
Capacitor

$$q_{out} = \frac{1}{KD}q_{in} = \frac{1}{K}\int_0^t q_{in}\,dt + \left(q_{out}\right)_{initial}$$

Step Response Inductor

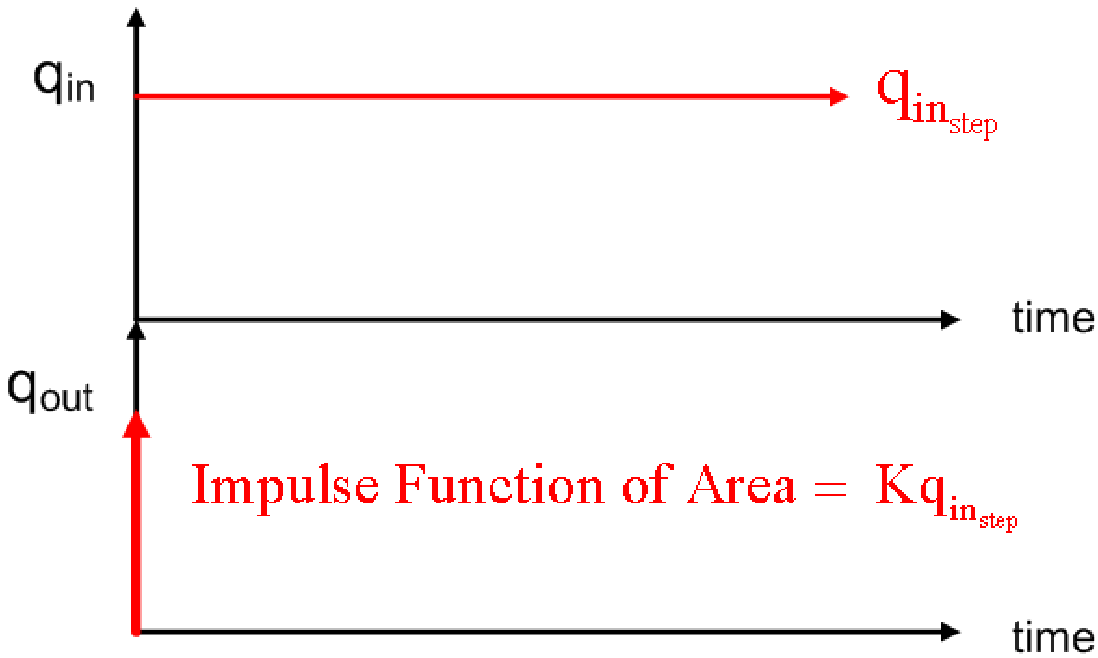
$$e = L\frac{di}{dt} = LDi$$

$$e = LDi$$



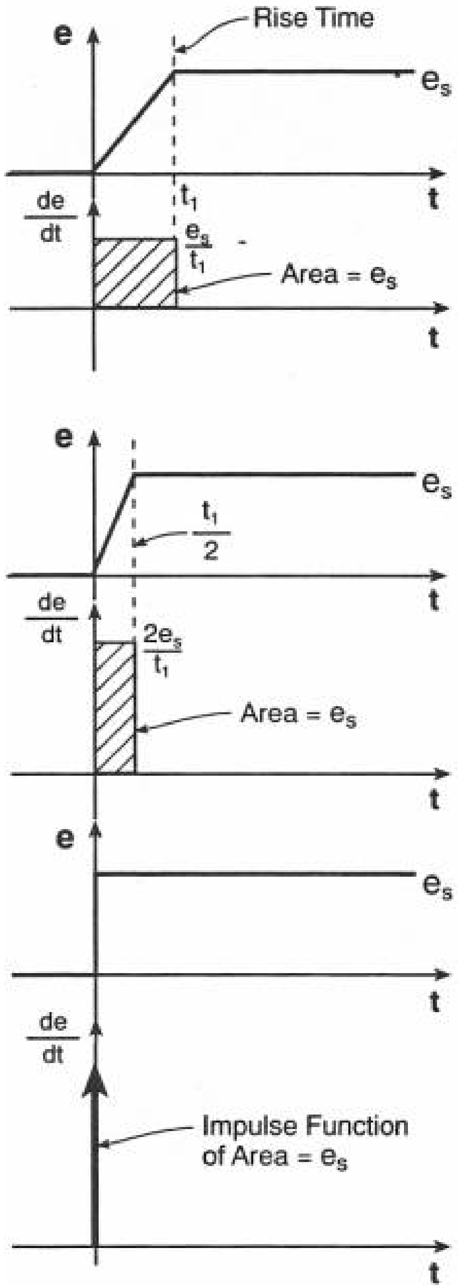
$$q_{out} = KDq_{in} = K\frac{dq_{in}}{dt}$$

$q_{in} = i$
$q_{out} = e$



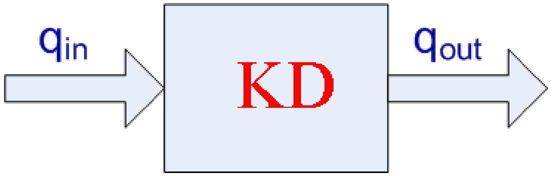
- Step Response and Impulse Response
  - By a step input of any variable we mean a situation where the system is "at rest" at time $t = 0$ and we instantly change the input quantity, from wherever it was just before $t = 0$, by a given amount, either positive or negative, and then keep the input constant at this new value "forever."
  - The integral of a step input is a ramp and the derivative of a step input is an impulse.

The impulse function is explained by the figure, where we approximate the step function by a terminated ramp and then let the rise time of the ramp approach zero. As we let the ramp get steeper and steeper, the magnitude of *de/dt* approaches infinity, and its duration approaches zero, but the area under it will always be e_s. If $e_s = 1$ (a unit step function), its derivative is called the unit impulse function with an area or strength equal to one unit. The step function is the integral of the impulse function, or conversely, the impulse function is the derivative of the step function. When we multiply the impulse function by some number, we increase the "strength of the impulse", but "strength" now means area, not height as it does for "ordinary" functions.



- An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration is very short compared to the system's speed of response, then we can approximate the pulse input by an impulse function. The impulse input supplies energy to the system in an infinitesimal time.
- The step response of a component or system is the time response to a step input of some magnitude. The impulse response of a system is the derivative of the step response and is the time response to an impulse input of some strength.

Frequency Response (Steady-State)



$$q_{out} = KDq_{in} = K\frac{dq_{in}}{dt}$$

$$e = L\frac{di}{dt} = LDi$$

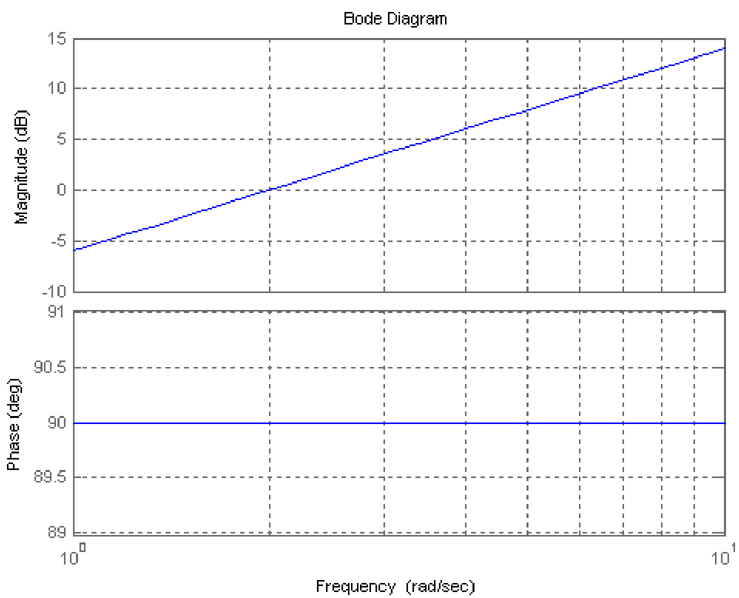
$$q_{in} = A\sin(\omega t)$$

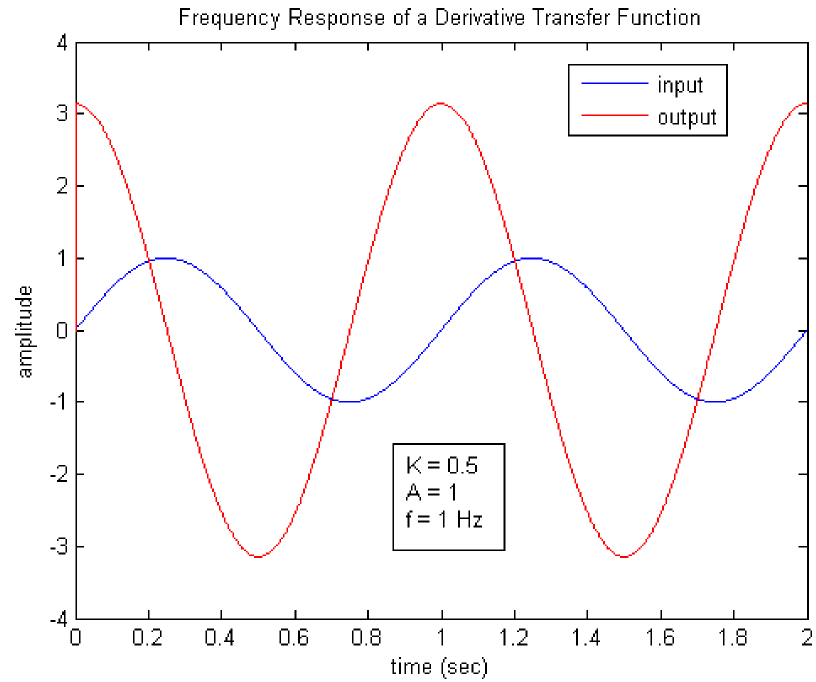
$$e = LDi$$

$$q_{out} = KA\omega\cos(\omega t) = KA\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

$q_{in} = i$

$q_{out} = e$





Inductor

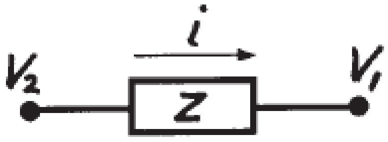
- Use of Inductors
  - The use of inductors is rapidly disappearing from electronics.
  - There are two main types of inductors: large, massive inductors to be used in power supplies and small inductors used in low-power, frequency-discriminating circuitry.
  - For the large inductors, the main parameters are: inductance (usually a few mH to 50 H), the nominal resistance of the winding, the maximum current (a few mA to several A, set by heating limits), the size (often large), and the cost.

- The small inductors often look like low-powered resistors, with inductances in the 0.1 μH to 100 mH range, and have color-code bands to indicate the value of the inductance. The low-value inductors are air-core coils, whereas the larger-value inductors are made with ferrite cores.
- Other small inductors in the 10 to 1000 μH range look like small tire-shaped windings typically of the order of an inch or less in diameter usually mounted on some sort of a cylindrical core. These coils can come shielded for situations where noise radiation or noise pickup may be a problem.
- For small inductors the interesting facts are the inductance (usually in μH), the resistance of the coil, the current-carrying limits (usually in the mA range), the energy loss as a function of frequency, some indication of the maximum (or minimum) frequency at which the coil is intended to be used, the size, and a short description of the type of coil.

Electrical Impedance

- A variable that flows through an element is impeded by the element.
- Electrical impedance is a generalization of the simple voltage/current relation called resistance for resistors.
- It can be applied to capacitors, inductors, and to entire circuits.
- It assumes ideal (linear) behavior of the device.
- Electrical impedance is defined as the transfer function relating voltage and current:

$$Z(D) = \frac{e}{i}(D)$$



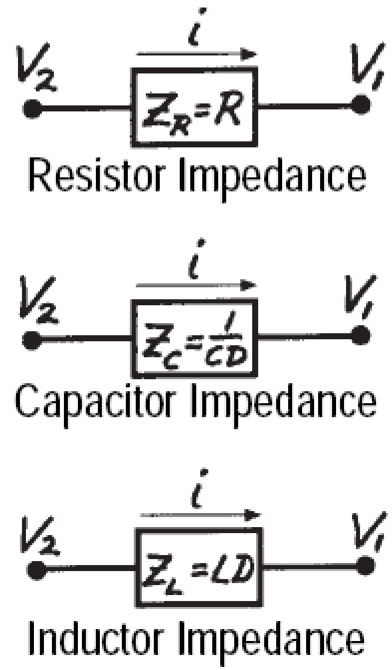
Note: Z is a function of the differential operator D, as is e/i.

– The impedances for the pure/ideal electrical elements are:

$$Z_R(D) = R$$

$$Z_C(D) = \frac{1}{CD}$$

$$Z_L(D) = LD$$



– Impedance is most useful in characterizing the dynamic behavior of components and systems.

- The impedances for the pure/ideal electrical elements are:

$$Z_R(D) = R \qquad Z_R(i\omega) = R$$

$$Z_C(D) = \frac{1}{CD} \qquad Z_C(i\omega) = \frac{1}{i\omega C}$$

$$Z_L(D) = LD \qquad Z_L(i\omega) = i\omega L$$

- The impedances for the pure/ideal mechanical elements are:

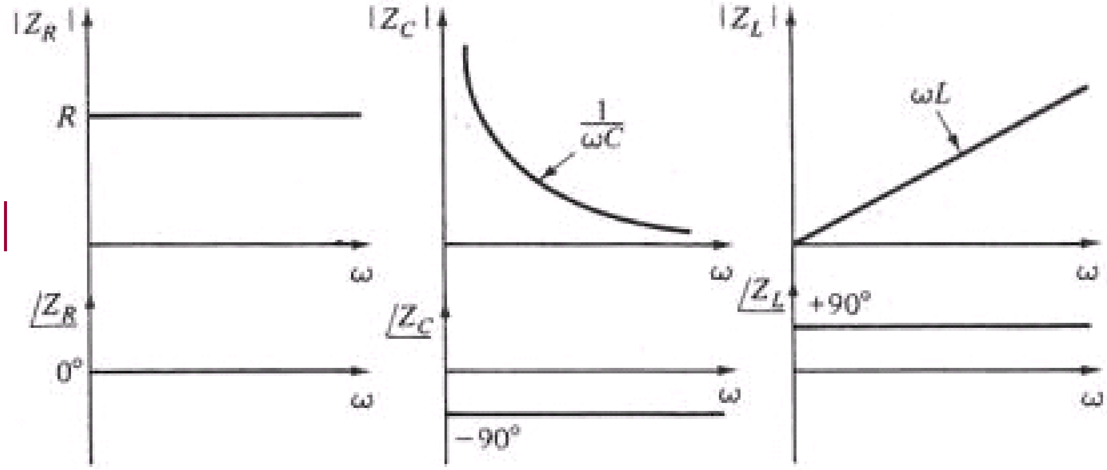
$$Z_B(D) = \frac{f}{v}(D) = B \qquad Z_B(i\omega) = B$$

$$Z_S(D) = \frac{f}{v}(D) = \frac{1}{C_S D} \qquad Z_S(i\omega) = \frac{1}{i\omega C_S}$$

$$Z_M(D) = \frac{f}{v}(D) = MD \qquad Z_M(i\omega) = i\omega M$$

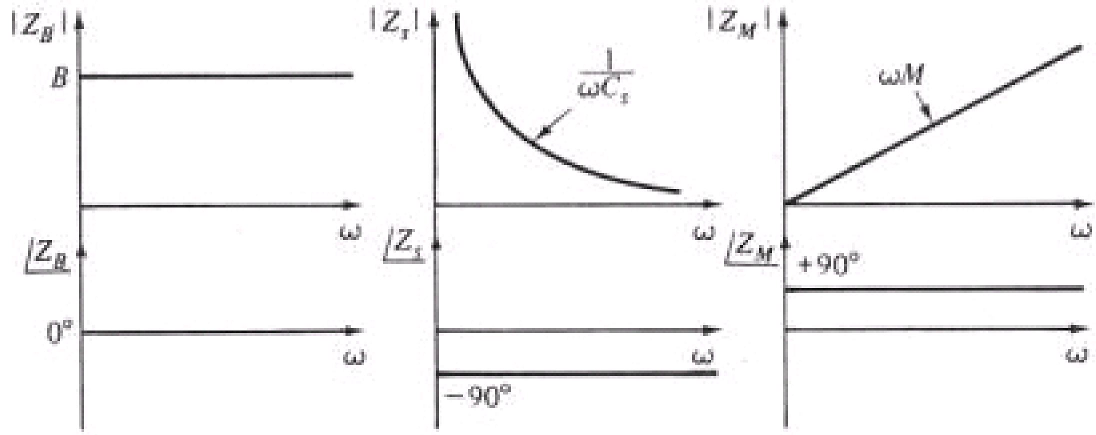
Impedances of Mechanical & Electrical Elements

force	⇔	voltage
velocity	⇔	current
damper	⇔	resistor
spring	⇔	capacitor
mass	⇔	inductor



Impedances of Electrical Elements

(a)



Impedances of Mechanical Elements

(b)

- It is also useful in the solution of routine circuit problems.

$$Z = \frac{A_e}{A_i} \angle \phi = M \angle \phi = M\cos\phi + iM\sin\phi = Z_R + iZ_X$$

$$R = Z_R = \text{resistive impedance}$$

$$X = Z_X = \text{reactive impedance}$$

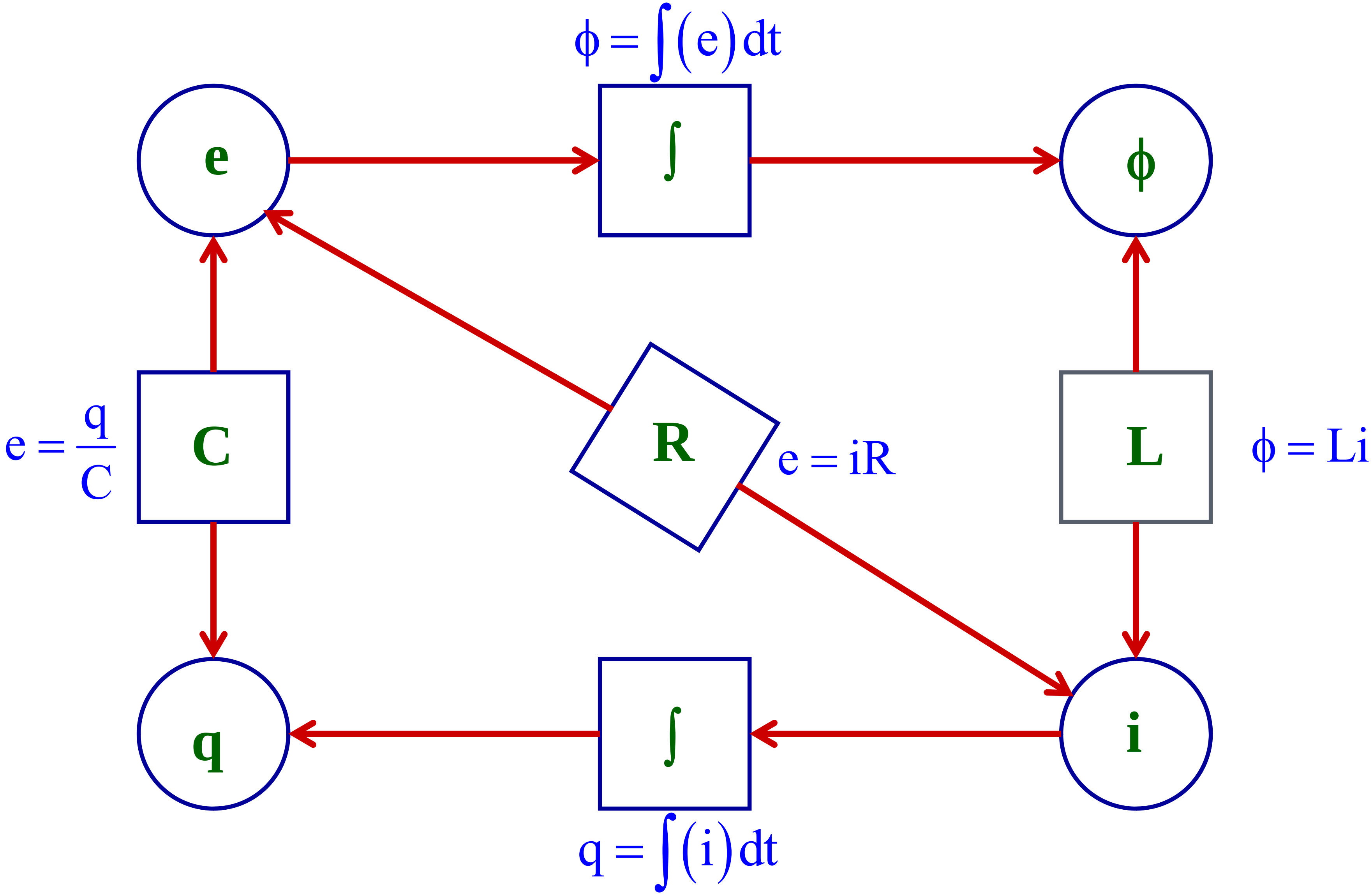
- If Z_X is a positive number, the reactive impedance is “behaving like an inductor” and is called inductive reactance; if negative, it is called capacitive reactance.

- Given R and X, one can always compute the magnitude and phase angle of the impedance:

$$M = \sqrt{R^2 + X^2} \qquad \phi = \tan^{-1}\left(\frac{X}{R}\right)$$

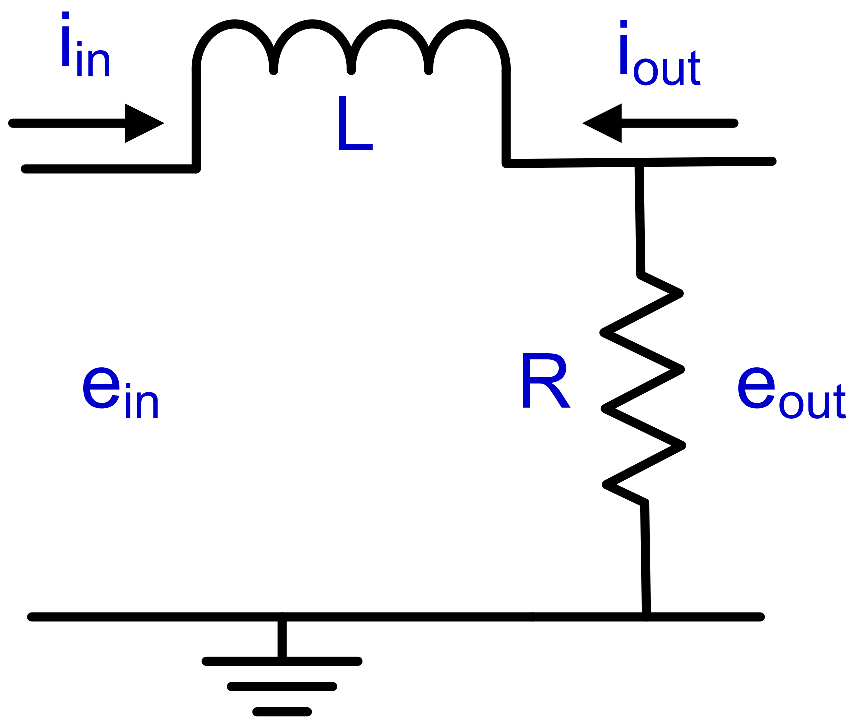
- Since the sinusoidal impedance gives the amplitude ratio and phase angle of voltage with respect to current, if the impedance of any circuit (no matter how complex) is known (from either theory or measurement), and either voltage or current is given, we can quickly calculate the other.

- The rules for combining series or parallel impedances are extensions to the dynamic case of the rules governing series and parallel static resistance elements.
  - If the same flow passes through two or more impedances, those impedances are said to be in *series*, and they are equivalent to a single impedance whose impedance is the sum of the individual impedances.
  - If the same effort difference exists across two or more impedances, those impedances are said to be in *parallel* and they are equivalent to a single impedance whose reciprocal is equal to the sum of the reciprocals of the individual impedances.



General Model Structure for Electrical Systems

Physical System for Investigation

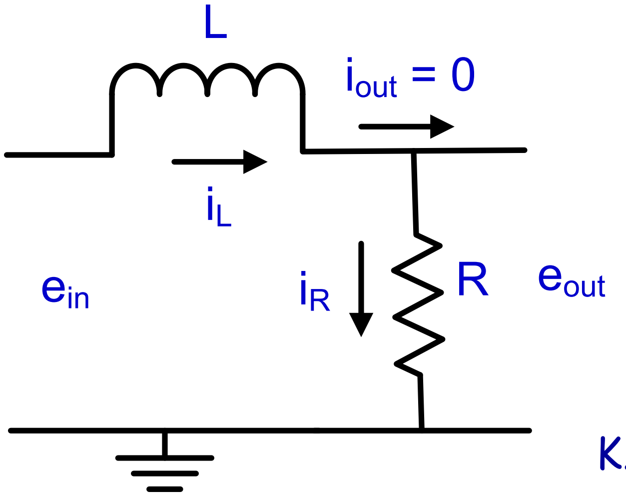


LR Circuit

Electrical System

Physical Modeling

- Simplifying Assumptions
  - Resistor is pure and ideal; Inductor is ideal but has resistance.
  - Voltage source is ideal and supplies the intended voltage to the circuit no matter how much current (and thus power) this might require.
  - Measuring device is ideal and does not load the circuit by drawing any current.

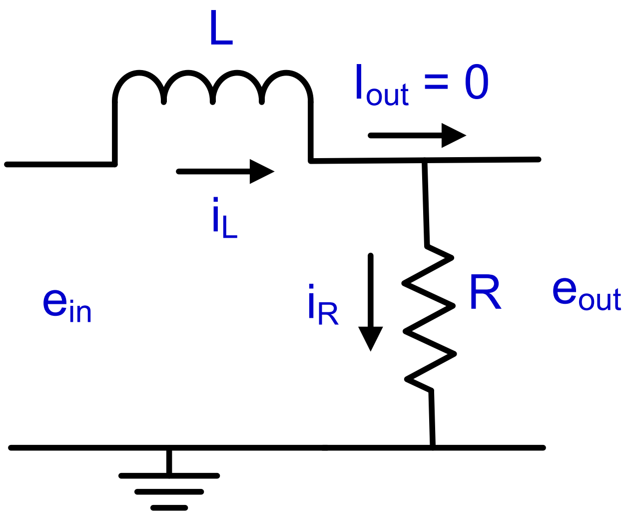


Model Parameter Identification

- Measure component values using the DMM.

- LR Circuit
  - $R = 56\ \Omega$
  - $L = 0.955$ mH nominal, $R_L = 3\ \Omega$

Mathematical Modeling of System



Basic Component Equations (Constitutive Equations)

$$e_{in} - e_{out} = L\frac{di_L}{dt}$$

$$e_{out} = i_R R$$

KVL $$e_{in} - L\frac{di_L}{dt} - e_{out} = 0$$

KCL $$i_L = i_R + i_{out} = i_R + 0$$

$$e_{in} - L\frac{d}{dt}\left(\frac{e_{out}}{R}\right) - e_{out} = 0$$

$$\frac{L}{R}\frac{de_{out}}{dt} + e_{out} = e_{in}$$

$$\frac{L}{R}\left(De_{out}\right) + e_{out} = e_{in}$$

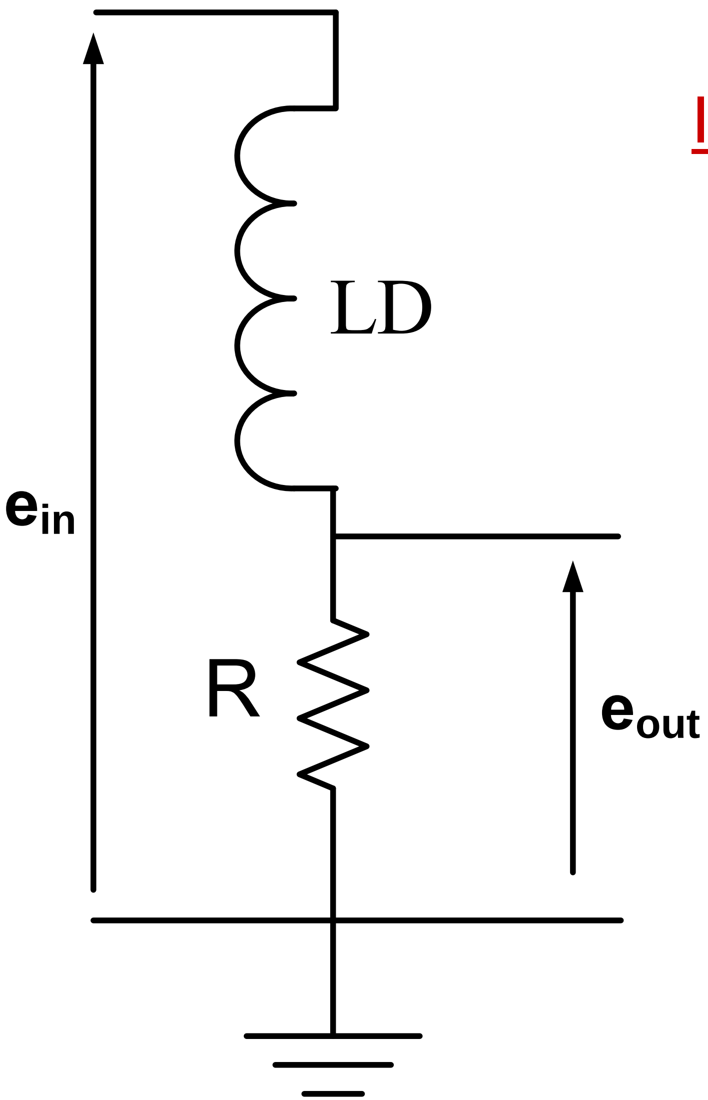
$$\left(\frac{L}{R}D + 1\right)e_{out} = e_{in}$$

$$\frac{e_{out}}{e_{in}} = \frac{1}{\frac{L}{R}D + 1} = \frac{K}{\tau D + 1}$$

$$K = 1$$

$$\tau = \frac{L}{R}$$

Another Approach: Impedance + Voltage Divider



Impedance:

$$e = iR \qquad \frac{e}{i} = R$$

$$e = L\frac{di}{dt} = LDi \qquad \frac{e}{i} = LD$$

$$\frac{e_{out}}{e_{in}} = \frac{R}{LD + R} = \frac{1}{\frac{L}{R}D + 1}$$

$$\left(\frac{L}{R}D + 1\right)e_{out} = (1)e_{in}$$

$$\left(\frac{L}{R}D\right)e_{out} + e_{out} = e_{in}$$

$$\frac{L}{R}\frac{de_{out}}{dt} + e_{out} = e_{in}$$

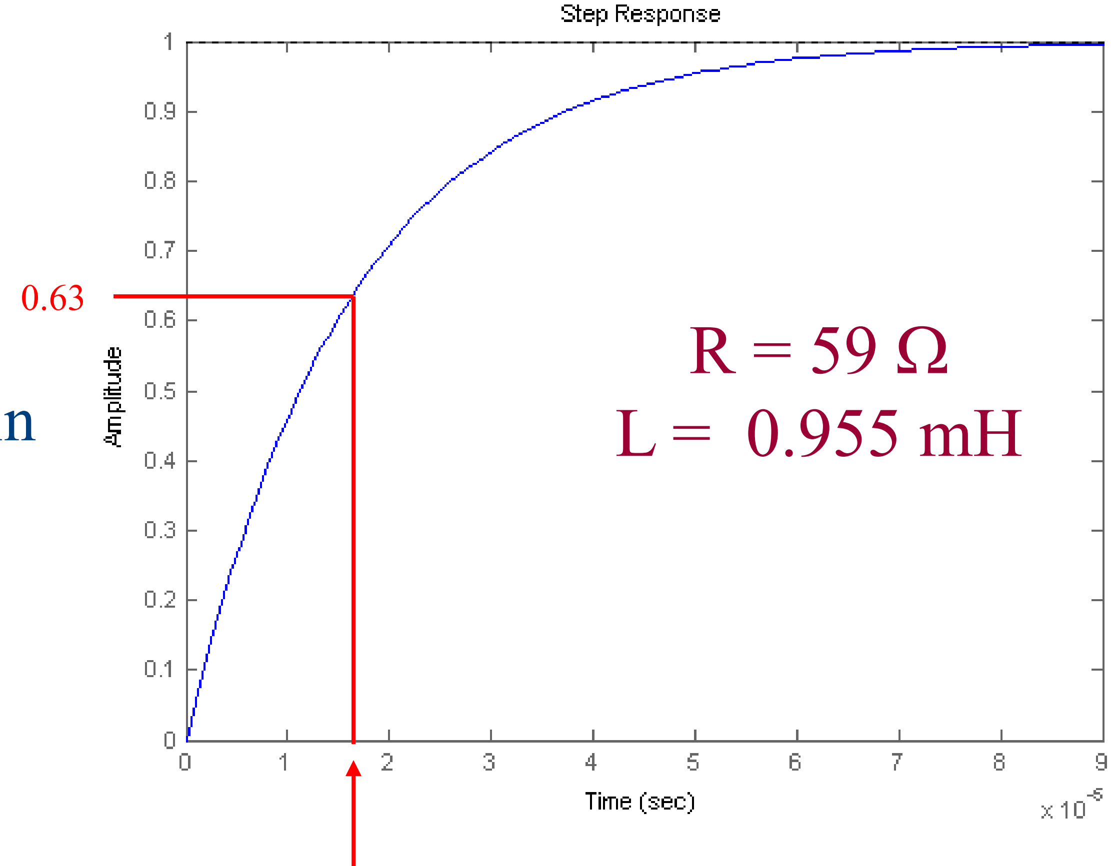
Mathematical Analysis and Prediction

LR Circuit Unit Step Response

$$\frac{e_{out}}{e_{in}} = \frac{1}{\frac{L}{R}D+1} = \frac{K}{\tau D+1}$$

$$K = 1 = \text{Steady-State Gain}$$

$$\tau = \frac{L}{R} = \text{Time Constant}$$



$$\tau = \frac{L}{R} = 0.0162 \text{ msec}$$

- Time Constant τ
  - Time it takes the step response to reach 63% of the steady-state value, Ke_{in}.
- Rise Time $T_r = 2.2\ \tau$
  - Time it takes the step response to go from 10% to 90% of the steady-state value, Ke_{in}.
- Delay Time $T_d = 0.69\ \tau$
  - Time it takes the step response to reach 50% of the steady-state value, Ke_{in}.
- Steady-State Value
  - The steady-state value of the response is Ke_{in} and at 4τ seconds (4 time constants), the response has reached 98% of the steady-state value; for all practical purposes, this is steady state.

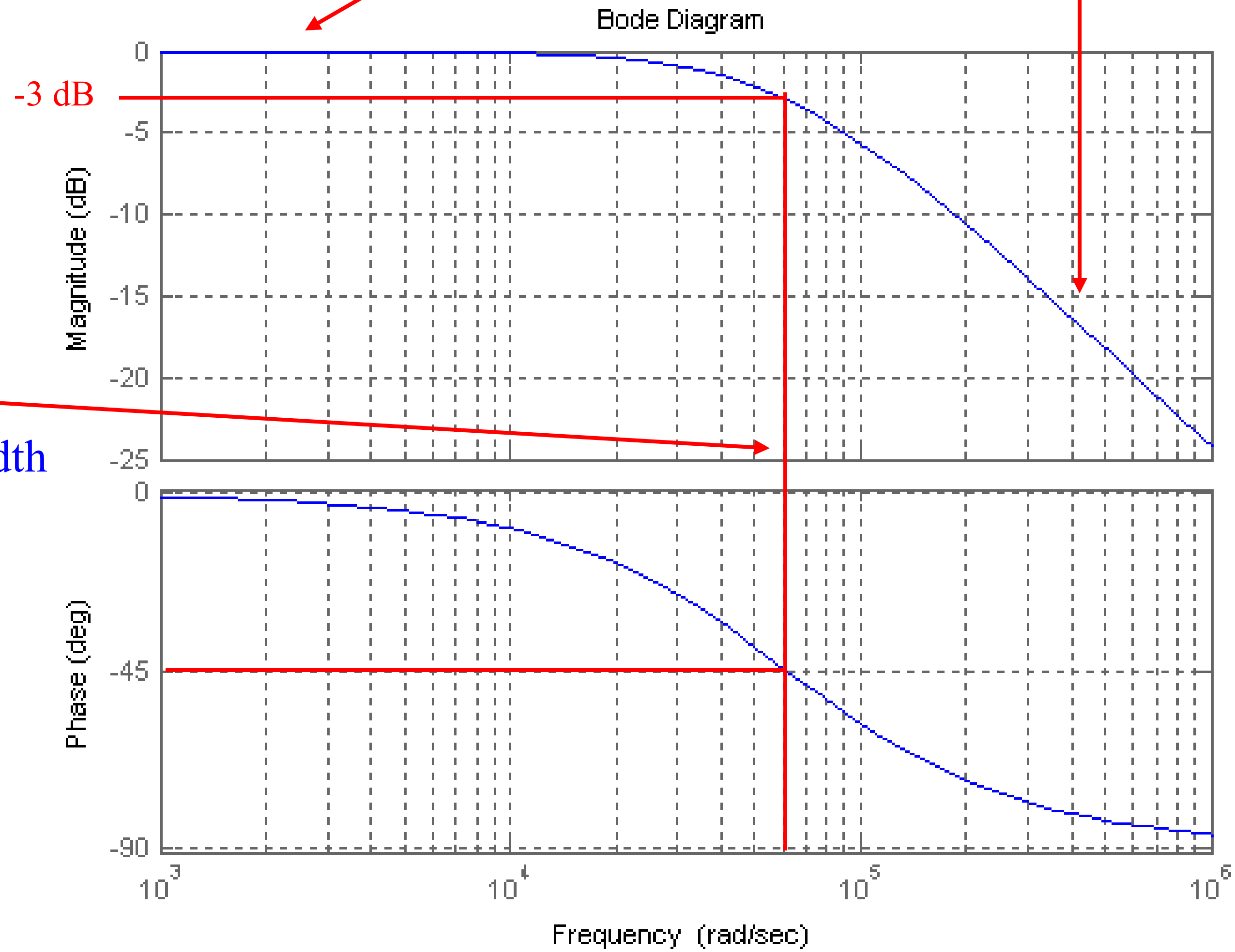
LR Circuit Frequency Response

DC Value = K

Slope = -20 dB/decade

$$\frac{1}{\tau} = \frac{R}{L} = 61780 \ \frac{\text{rad}}{\text{sec}}$$

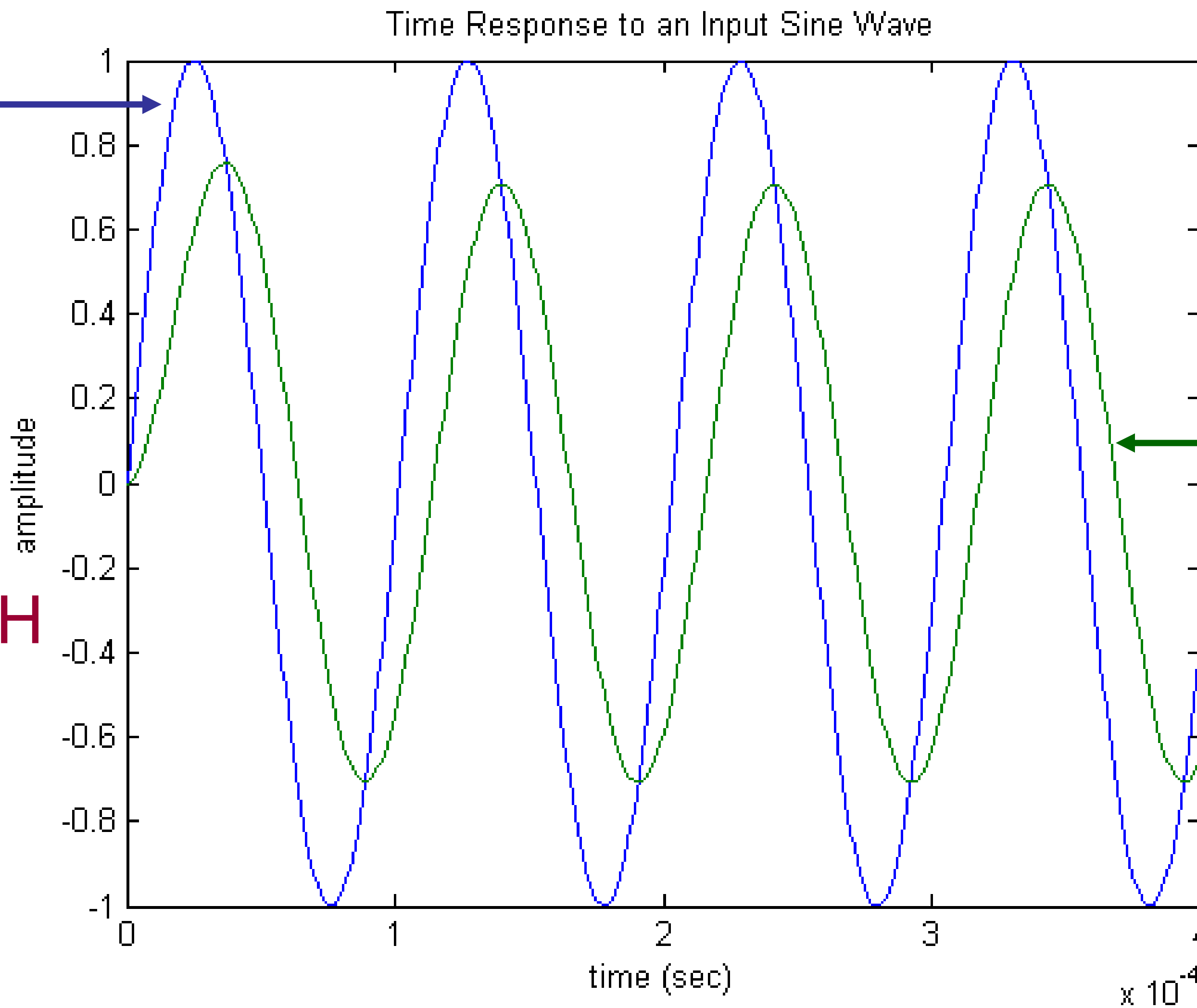
$$= 9832 \ \text{Hz} = \text{Bandwidth}$$



LR Circuit

Amplitude Ratio = 0.707 = -3 dB   Phase Angle = -45°

Input
9832 Hz
Sine Wave



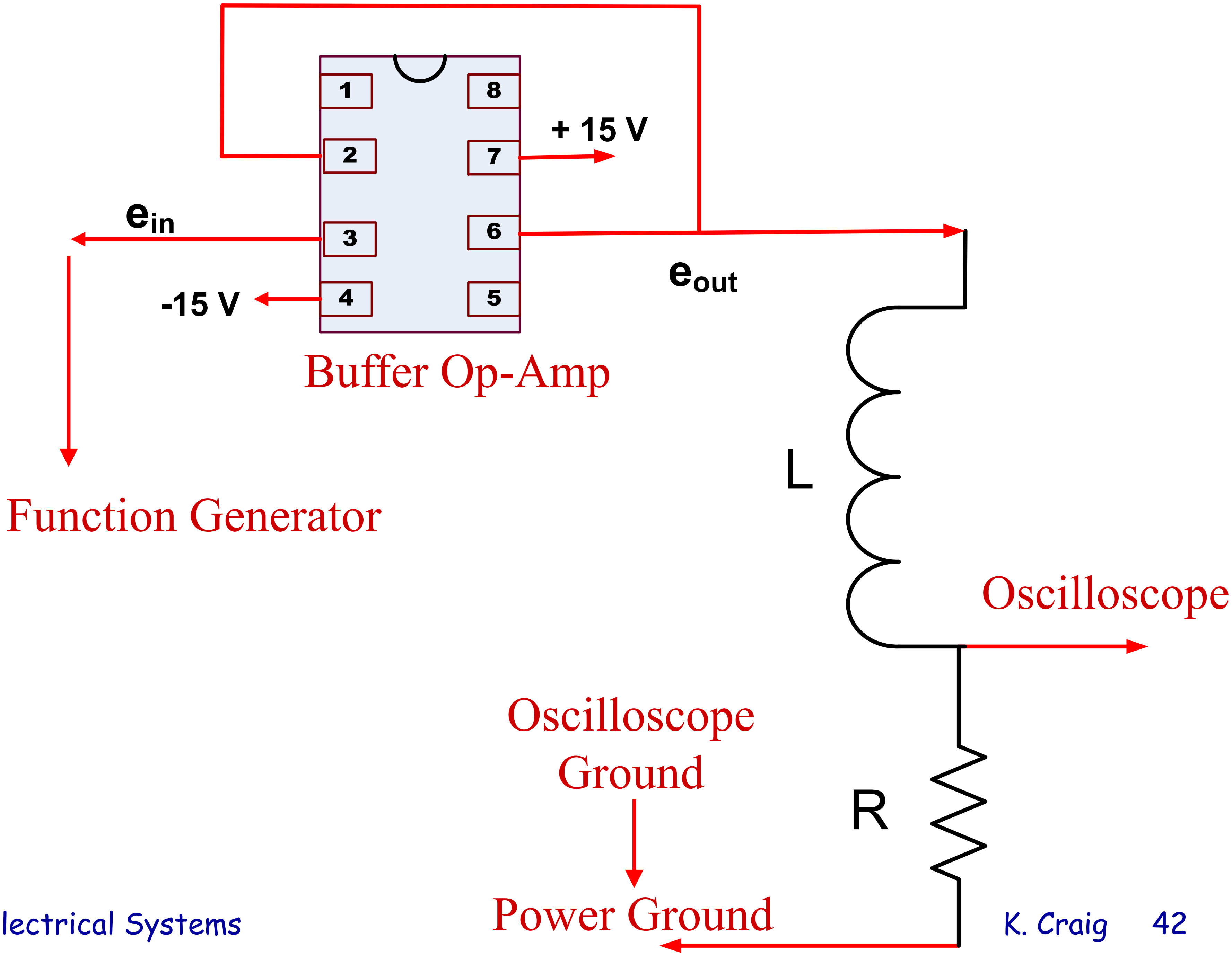
Output

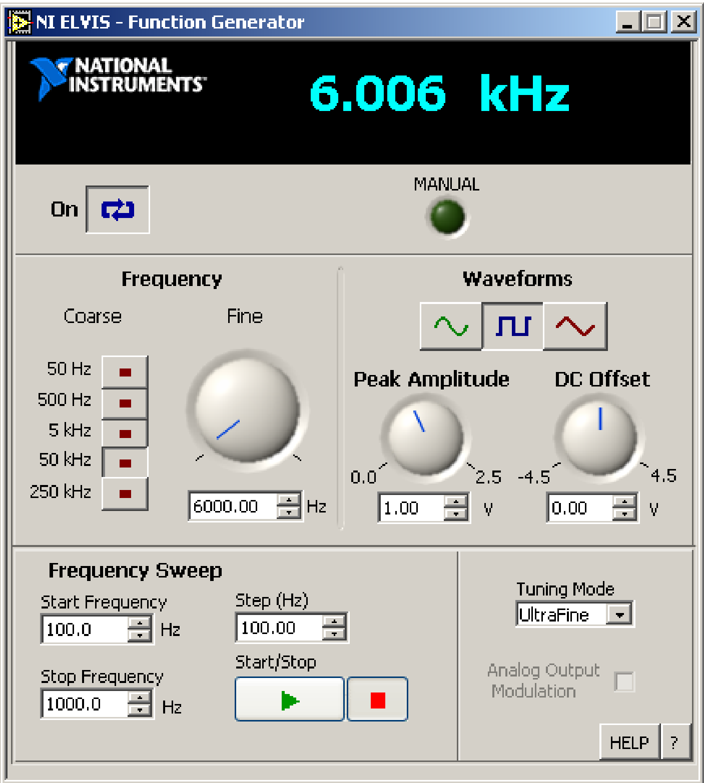
R = 59 Ω
L = 0.955 mH

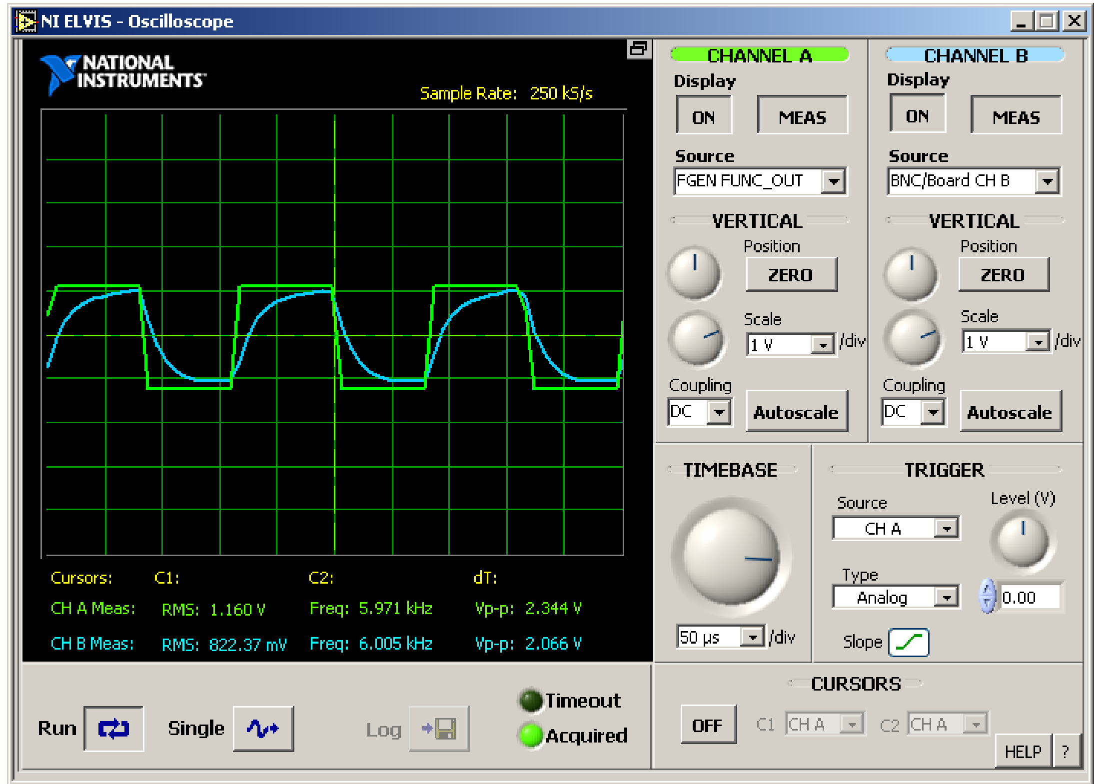
- Bandwidth
  - The bandwidth is the frequency where the amplitude ratio drops by a factor of 0.707 = -3dB of its gain at zero or low-frequency.
  - For a 1st-order system, the bandwidth is equal to $1/\tau$.
  - The larger (smaller) the bandwidth, the faster (slower) the step response.
  - Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.
- Note that the amplitude ratio follows low- and high-frequency asymptotes, i.e., slope 0 and –20 dB/decade, respectively, and the phase angle approaches -90° asymptotically. At the break frequency $1/\tau$, the phase angle is -45°.

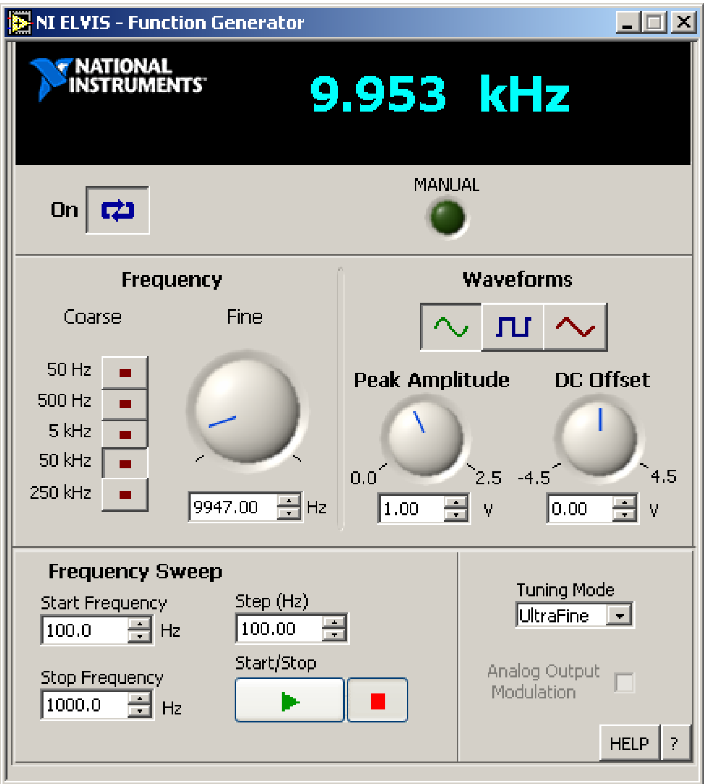
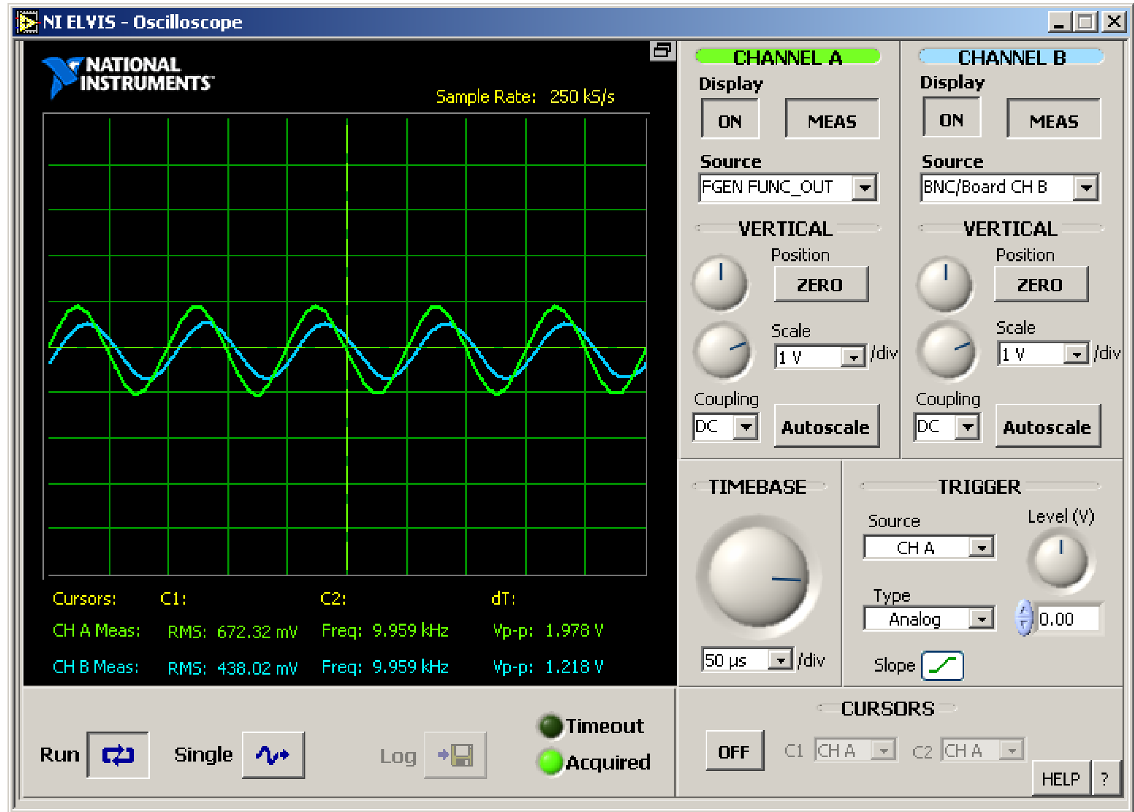
Measurements Using LabVIEW

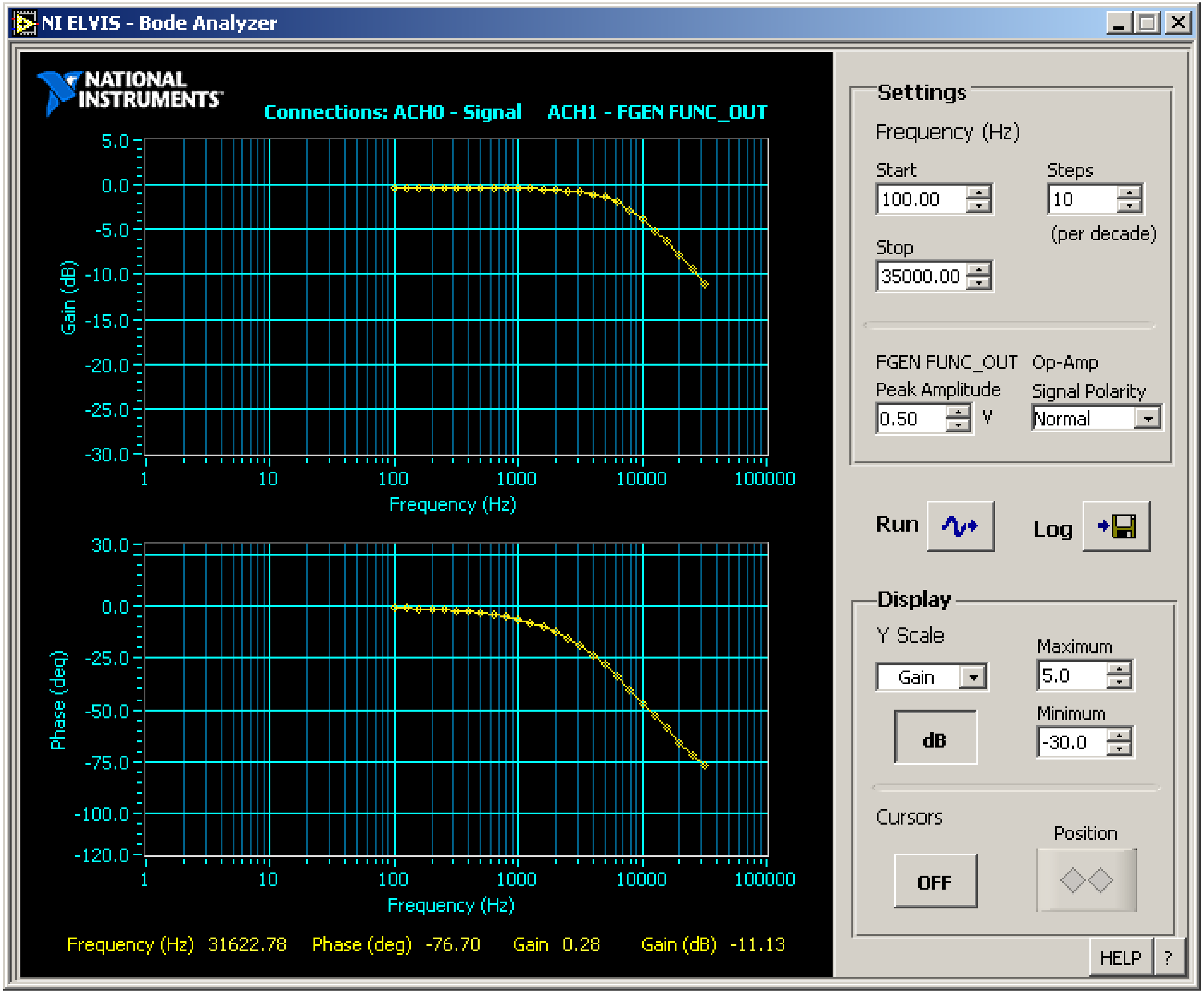
- LR Circuit
  - Step Response
  - Frequency Response


1
2
3
4
5
6
7
8
+ 15 V
-15 V
e_in
e_out
Buffer Op-Amp
Function Generator
L
R
Oscilloscope
Oscilloscope Ground
Power Ground

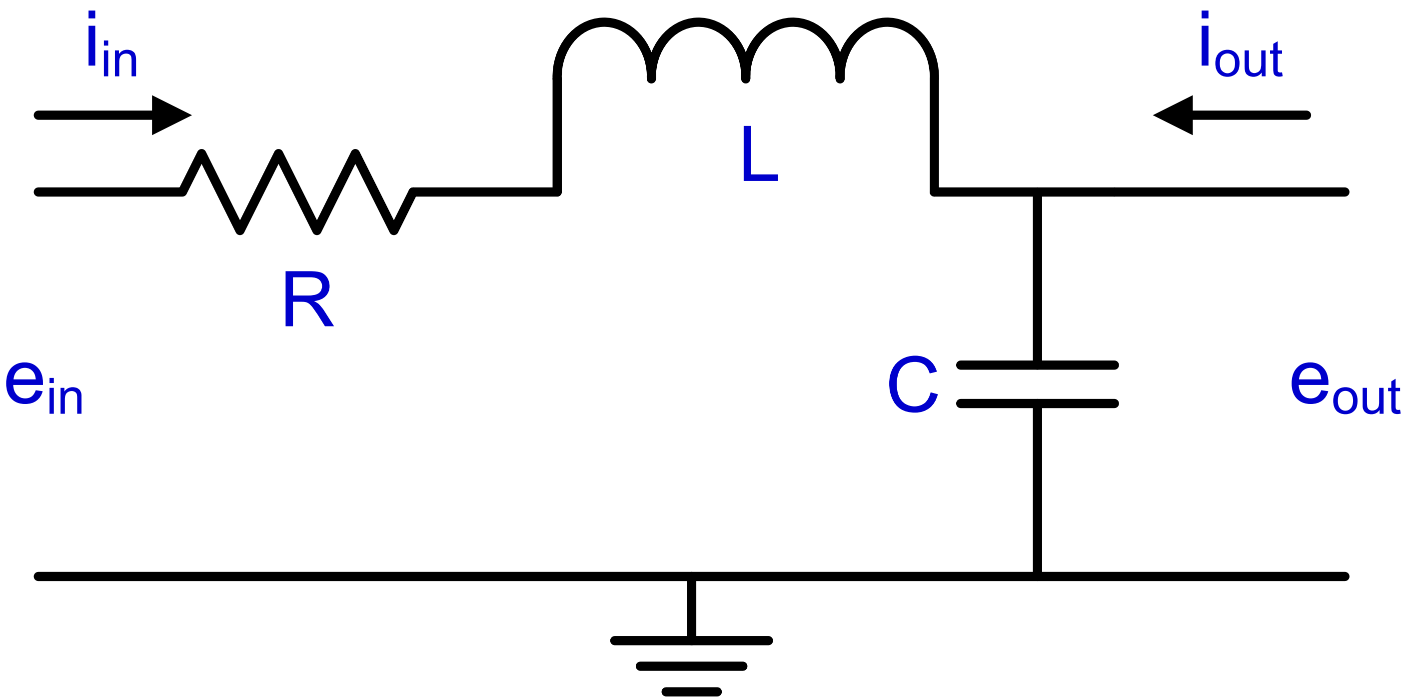

NI ELVIS - Function Generator
NATIONAL INSTRUMENTS
6.006 kHz
On
MANUAL
Frequency
Coarse
Fine
50 Hz
500 Hz
5 kHz
50 kHz
250 kHz
6000.00
Hz
Waveforms
Peak Amplitude
DC Offset
0.0
2.5
-4.5
4.5
1.00
V
0.00
V
Frequency Sweep
Start Frequency
100.0
Hz
Step (Hz)
100.00
Stop Frequency
1000.0
Hz
Start/Stop
Tuning Mode
UltraFine
Analog Output Modulation
HELP
?


NI ELVIS - Oscilloscope
NATIONAL INSTRUMENTS
Sample Rate: 250 kS/s
Cursors: C1: C2: dT:
CH A Meas: RMS: 1.160 V Freq: 5.971 kHz Vp-p: 2.344 V
CH B Meas: RMS: 822.37 mV Freq: 6.005 kHz Vp-p: 2.066 V
CHANNEL A
CHANNEL B
Display
ON
MEAS
Source
FGEN FUNC_OUT
BNC/Board CH B
VERTICAL
Position
ZERO
Scale
1 V
/div
Coupling
DC
Autoscale
TIMEBASE
50 µs
/div
TRIGGER
Source
CH A
Level (V)
Type
Analog
0.00
Slope
Run
Single
Log
Timeout
Acquired
CURSORS
OFF
C1
CH A
C2
CH A
HELP
?


NI ELVIS - Function Generator
NATIONAL INSTRUMENTS
9.953 kHz
On
MANUAL
Frequency
Coarse
Fine
50 Hz
500 Hz
5 kHz
50 kHz
250 kHz
9947.00
Hz
Waveforms
Peak Amplitude
DC Offset
0.0
2.5
-4.5
4.5
1.00
V
0.00
V
Frequency Sweep
Start Frequency
100.0
Hz
Step (Hz)
100.00
Stop Frequency
1000.0
Hz
Start/Stop
Tuning Mode
UltraFine
Analog Output Modulation
HELP
?

NI ELVIS - Oscilloscope
NATIONAL INSTRUMENTS
Sample Rate: 250 kS/s
Cursors: C1: C2: dT:
CH A Meas: RMS: 672.32 mV Freq: 9.959 kHz Vp-p: 1.978 V
CH B Meas: RMS: 438.02 mV Freq: 9.959 kHz Vp-p: 1.218 V
CHANNEL A
Display
ON
MEAS
Source
FGEN FUNC_OUT
VERTICAL
Position
ZERO
Scale
1 V
/div
Coupling
DC
Autoscale
CHANNEL B
Display
ON
MEAS
Source
BNC/Board CH B
VERTICAL
Position
ZERO
Scale
1 V
/div
Coupling
DC
Autoscale
TIMEBASE
50 µs
/div
TRIGGER
Source
CH A
Level (V)
Type
Analog
0.00
Slope
Run
Single
Log
Timeout
Acquired
CURSORS
OFF
C1
CH A
C2
CH A
HELP
?


NI ELVIS - Bode Analyzer
NATIONAL INSTRUMENTS
Connections: ACH0 - Signal ACH1 - FGEN FUNC_OUT
Gain (dB)
5.0
0.0
-5.0
-10.0
-15.0
-20.0
-25.0
-30.0
1
10
100
1000
10000
100000
Frequency (Hz)
Phase (deg)
30.0
0.0
-25.0
-50.0
-75.0
-100.0
-120.0
Frequency (Hz) 31622.78 Phase (deg) -76.70 Gain 0.28 Gain (dB) -11.13
Settings
Frequency (Hz)
Start
100.00
Steps
10
(per decade)
Stop
35000.00
FGEN FUNC_OUT
Peak Amplitude
0.50
V
Op-Amp
Signal Polarity
Normal
Run
Log
Display
Y Scale
Gain
dB
Maximum
5.0
Minimum
-30.0
Cursors
OFF
Position
HELP
?

Physical System for Investigation

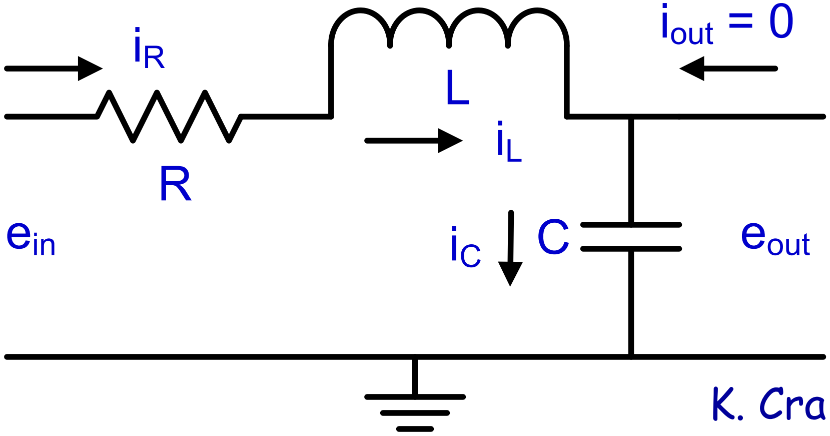


LRC Circuit

Electrical System

Physical Modeling

- Simplifying Assumptions
  - Resistor and Capacitor are pure and ideal; Inductor is ideal but has resistance
  - Voltage source is ideal and supplies the intended voltage to the circuit no matter how much current (and thus power) this might require
  - Measuring device is ideal and does not load the circuit by drawing any current

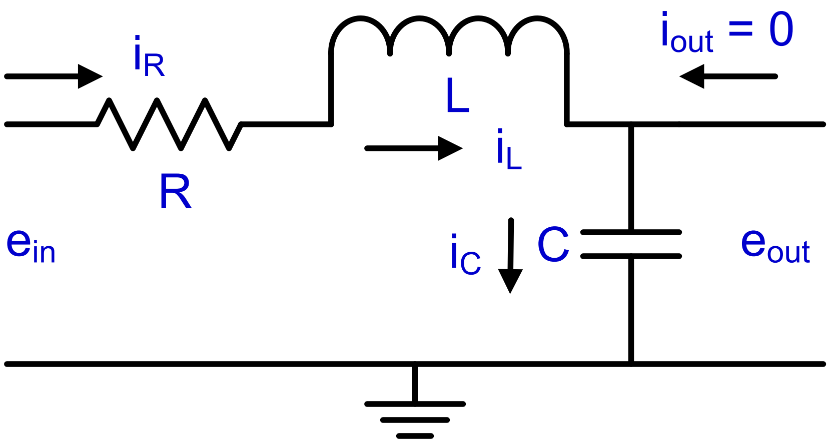


Model Parameter Identification

- Measure component values using the DMM.

- <u>LRC Circuit</u>
  - R = Potentiometer (500 Ω)
  - L = 22 mH (nominal)
  - C = 0.1 μF (nominal)

Mathematical Modeling of System



Basic Component Equations (Constitutive Equations)

$$e_L = L\frac{di_L}{dt}$$

$$e_R = i_R R$$

$$i_C = C\frac{de_C}{dt}$$

<u>KVL</u> $\quad e_{in} - e_R - e_L - e_C = 0$

<u>KCL</u> $\quad i_R = i_L = i_C \qquad i_{out} = 0$

$$e_{in} - Ri - L\frac{di}{dt} - e_{out} = 0$$

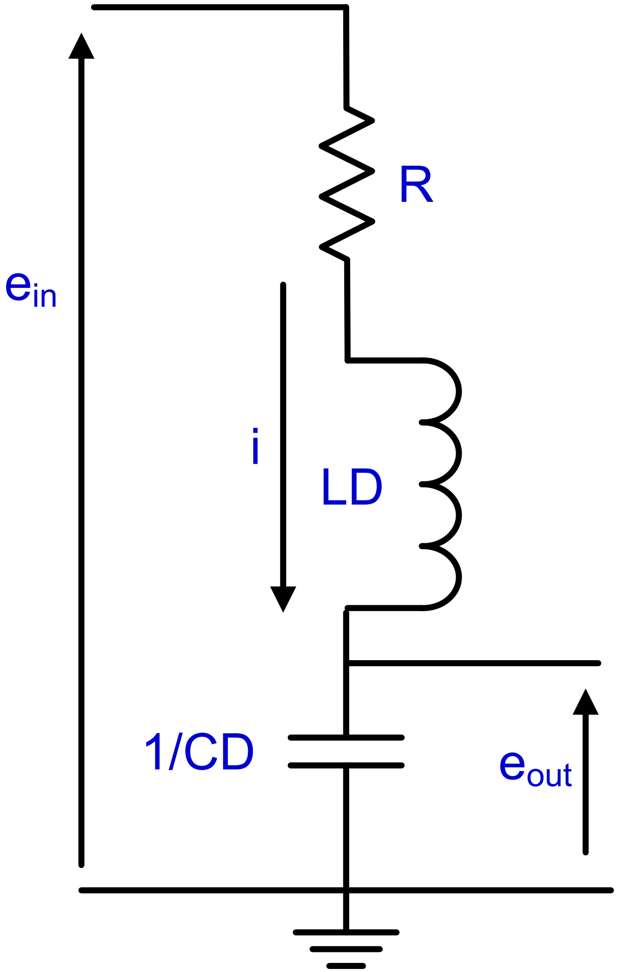
$$e_{in} - R\left(C\frac{de_{out}}{dt}\right) - L\frac{d}{dt}\left(C\frac{de_{out}}{dt}\right) - e_{out} = 0$$

$$LC\frac{d^2e_{out}}{dt^2} + RC\frac{de_{out}}{dt} + e_{out} = e_{in}$$

$$\left(LCD^2 + RCD + 1\right)e_{out} = e_{in}$$

$$\frac{e_{out}}{e_{in}} = \frac{1}{LCD^2 + RCD + 1}$$

Another Approach: Impedance + Voltage Divider



Impedance:

$$e = iR \qquad \frac{e}{i} = R$$

$$e = L\frac{di}{dt} = LDi \qquad \frac{e}{i} = LD$$

$$i = C\frac{de}{dt} = CDe \qquad \frac{e}{i} = \frac{1}{CD}$$

$$\frac{e_{out}}{e_{in}} = \frac{\frac{1}{CD}}{R + LD + \frac{1}{CD}} = \frac{1}{LCD^2 + RCD + 1}$$

$$\left(LCD^2 + RCD + 1\right)e_{out} = e_{in}$$

$$LC\frac{d^2e_{out}}{dt^2} + RC\frac{de_{out}}{dt} + e_{out} = e_{in}$$

Mathematical Analysis and Prediction

$$LC\frac{d^2e_{out}}{dt^2} + RC\frac{de_{out}}{dt} + e_{out} = e_{in}$$

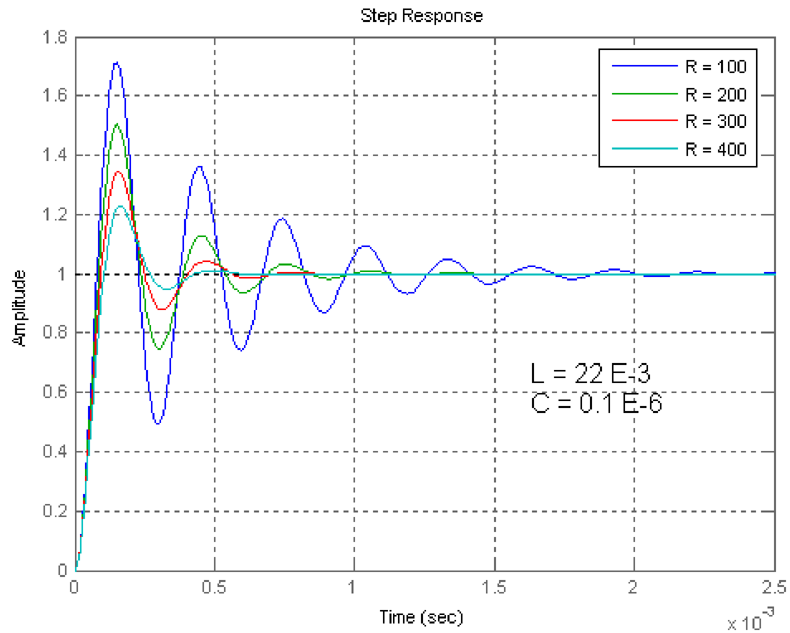
Differential Equation

$$\frac{e_{out}}{e_{in}} = \frac{1}{LCD^2 + RCD + 1}$$

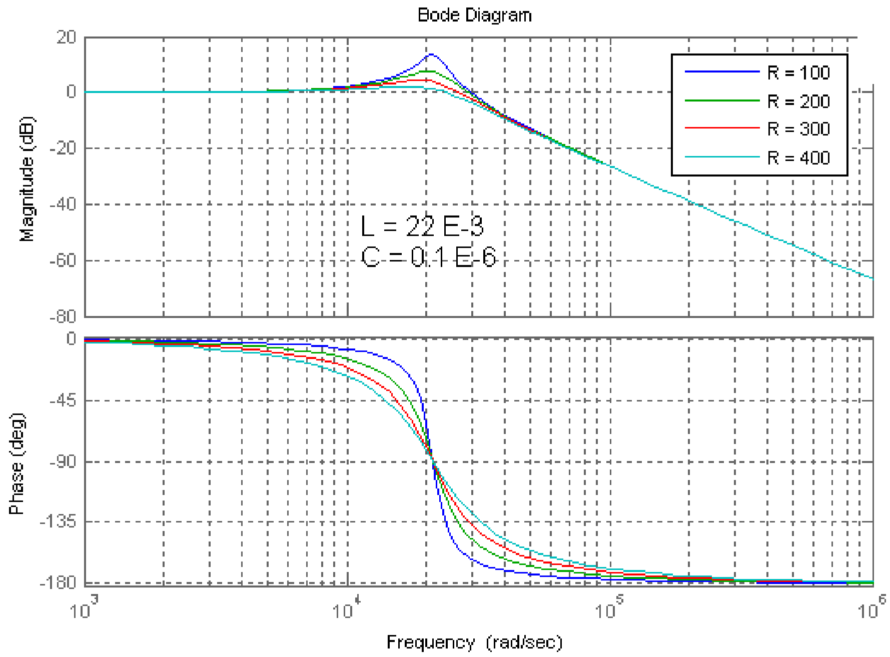
Transfer Function

L = 22 mH   C = 0.1 μF   R = 100, 200, 300, 400 Ω

MatLab Step and Frequency Response Plots



Unit Step Response LRC Circuit



Frequency Response LRC Circuit

2^{nd}-Order Dynamic System Model

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

$$\frac{1}{\omega_n^2} \frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_0}{dt} + q_0 = K_S q_i$$

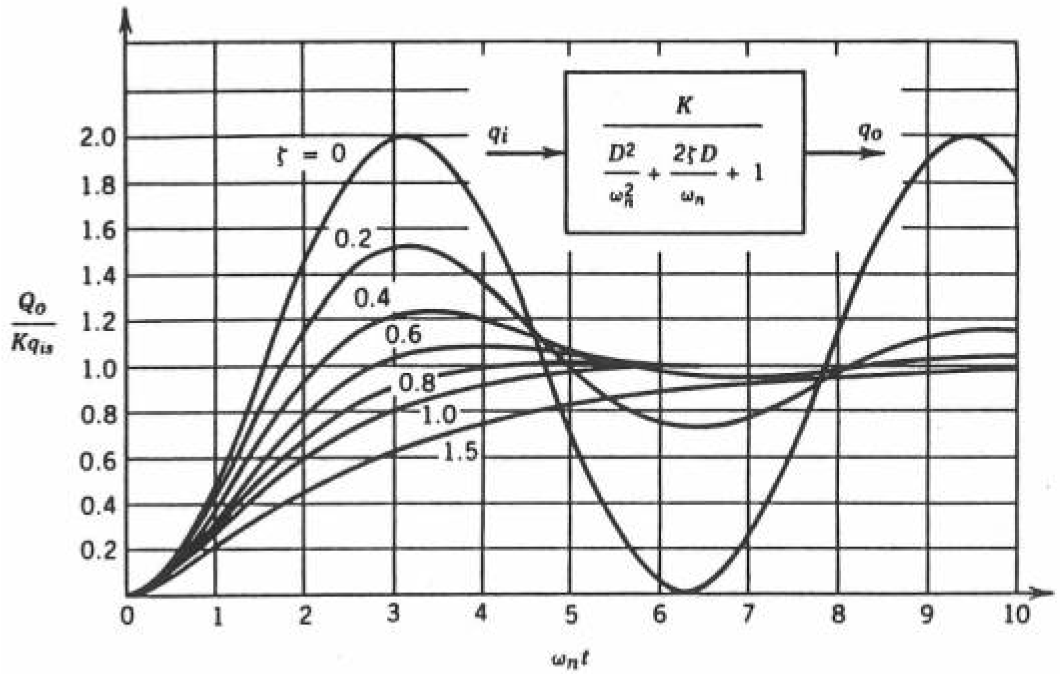
$$\frac{q_o}{q_i} = \frac{K_S}{\frac{1}{\omega_n^2} D^2 + \frac{2\zeta}{\omega_n} D + 1}$$

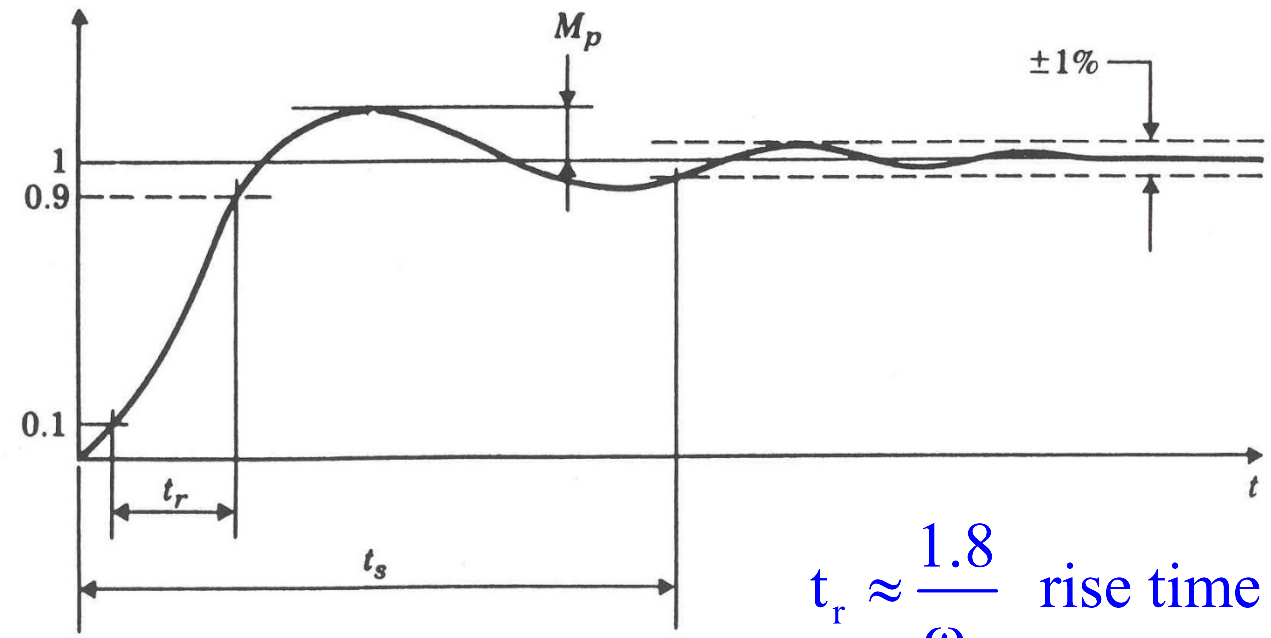
$$\omega_n = \sqrt{\frac{a_0}{a_2}} = \text{undamped natural frequency}$$

$$\zeta = \frac{a_1}{2\sqrt{a_2 a_0}} = \text{damping ratio}$$

$$K_S = \frac{b_0}{a_0} = \text{steady-state gain}$$

Step Response
of a
2^{nd}-Order System





Unit Step Response of a 2nd – Order Dynamic System

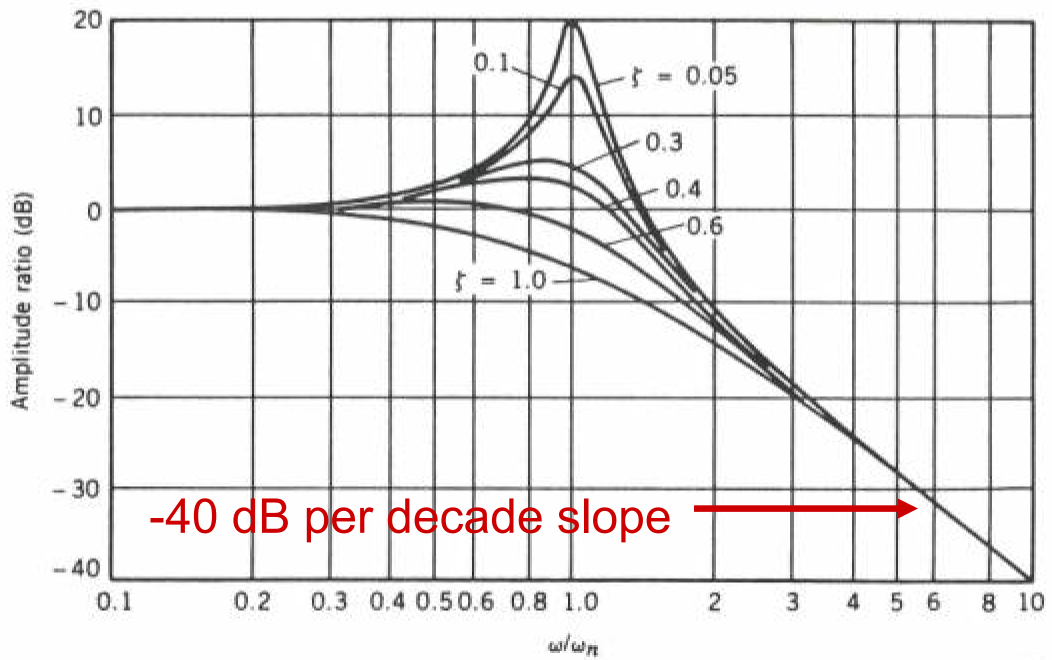
$$t_r \approx \frac{1.8}{\omega_n} \quad \text{rise time}$$

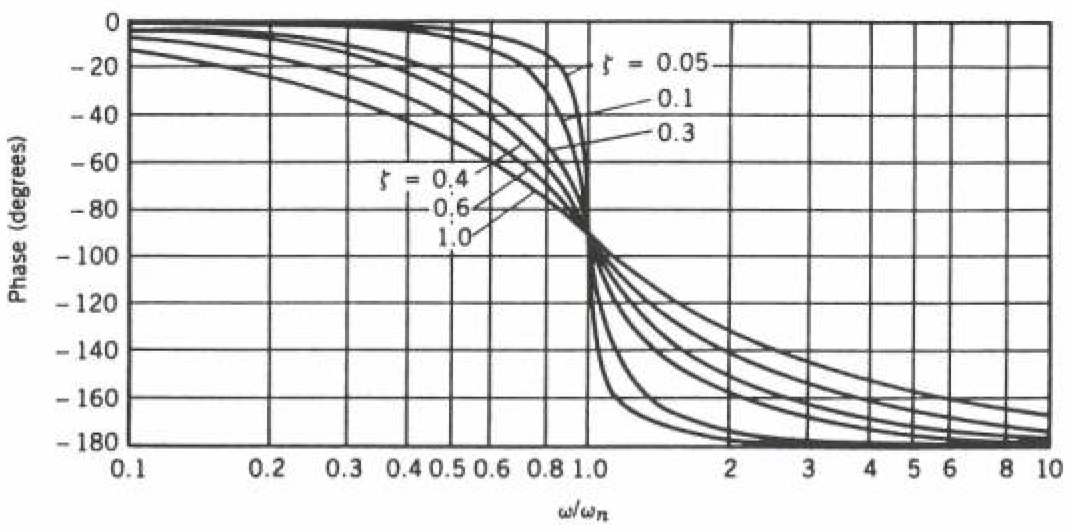
$$t_s \approx \frac{4.6}{\varsigma\omega_n} \quad \text{settling time}$$

$$M_p = e^{\frac{-\pi\varsigma}{\sqrt{1-\varsigma^2}}} \quad \left(0 \le \zeta < 1\right) \quad \text{overshoot}$$

$$\approx \left(1 - \frac{\zeta}{0.6}\right) \quad \left(0 \le \zeta \le 0.6\right)$$

Frequency Response of a 2^{nd} – Order Dynamic System





Some Observations

- When a physical system exhibits a natural oscillatory behavior, a 1st-order model (or even a cascade of several 1st-order models) cannot provide the desired response. The simplest model that does possess that possibility is the 2nd-order dynamic system model.
- This system is very important in control design.
  - System specifications are often given assuming that the system is 2nd order.
  - For higher-order systems, we can often approximate the system with a 2nd-order transfer function.

- Damping ratio ζ clearly controls oscillation; $\zeta < 1$ is required for oscillatory behavior.
- The undamped case ($\zeta = 0$) is not physically realizable (total absence of energy loss effects) but gives us, mathematically, a sustained oscillation at frequency ω_n.
- Natural oscillations of damped systems are at the damped natural frequency ω_d, and not at ω_n.

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- In hardware design, an optimum value of $\zeta = 0.64$ is often used to give maximum response speed without excessive oscillation.
- At $\omega = \omega_n$, the phase angle is exactly -90^o.

- Undamped natural frequency ω_n is the major factor in response speed. For a given ζ response speed is directly proportional to ω_n.
- Thus, when 2nd-order components are used in feedback system design, large values of ω_n (small time lags) are desirable since they allow the use of larger control effort before stability limits are encountered.
- For frequency response, a resonant peak occurs for $\zeta < 0.707$. The peak frequency is ω_p and the peak amplitude ratio depends only on ζ.

$$\omega_p = \omega_n \sqrt{1-2\zeta^2} \qquad \text{peak amplitude ratio} = \frac{K_S}{2\zeta\sqrt{1-\zeta^2}}$$

- Bandwidth
  - The bandwidth is the frequency where the amplitude ratio drops by a factor of 0.707 = -3dB of its gain at zero or low-frequency.
  - For a 1st -order system, the bandwidth is equal to $1/\tau$.
  - The larger (smaller) the bandwidth, the faster (slower) the step response.
  - Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.
  - For a 2nd-order system:

$$\mathrm{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

  - As ζ varies from 0 to 1, BW varies from $1.55\omega_n$ to $0.64\omega_n$. For a value of $\zeta = 0.707$, BW = ω_n. For most design considerations, we assume that the bandwidth of a 2nd-order system can be approximated by ω_n.

Summary

Transfer Function

$$\frac{e_{out}}{e_{in}} = \frac{1}{LCD^2 + RCD + 1}$$

$$= \frac{K_S}{\frac{1}{\omega_n^2}D^2 + \frac{2\zeta}{\omega_n}D + 1}$$

Hardware Parameters → Model Parameters

$$\omega_n = \sqrt{\frac{1}{LC}} \quad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \quad K_S = 1$$

$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_s \approx \frac{4.6}{\zeta\omega_n}$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Time Domain Performance

Frequency Domain Performance

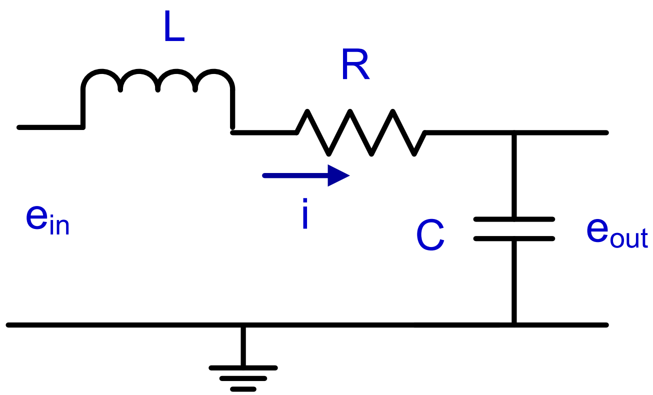
$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$

$$\text{peak amplitude ratio} = \frac{K_S}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$BW = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$$

Unit Step Response

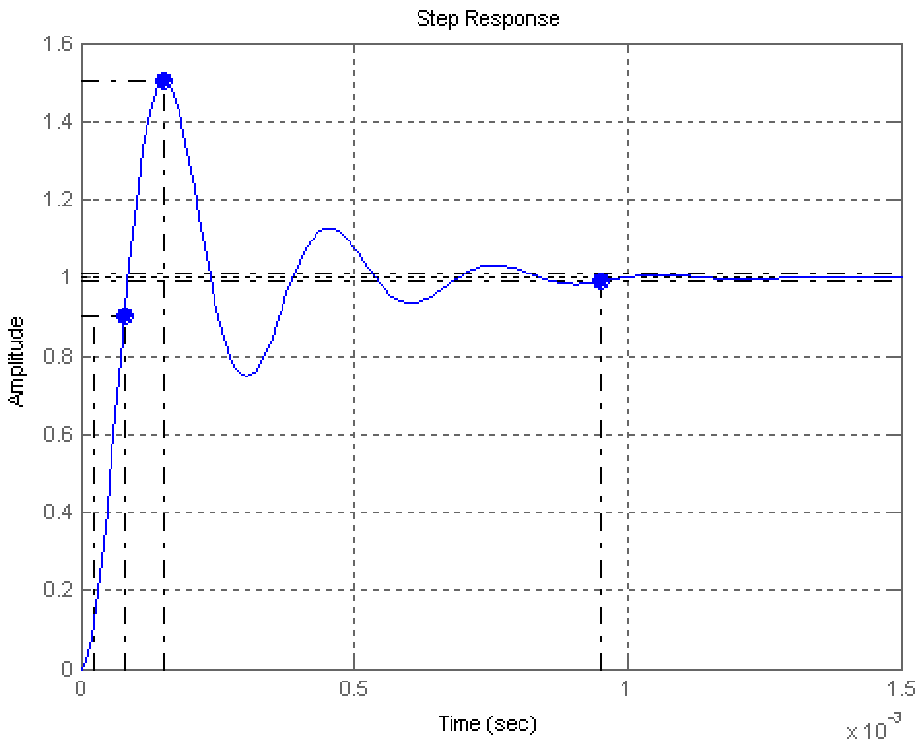


$$R = 200\ \Omega$$
$$C = 0.1\ \mu F$$
$$L = 22\ mH$$

$$\omega_n = \sqrt{\frac{1}{LC}} = 21320\ \text{rad/sec}$$
$$= 3393\ \text{Hz}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 0.213$$

$$K_S = 1$$

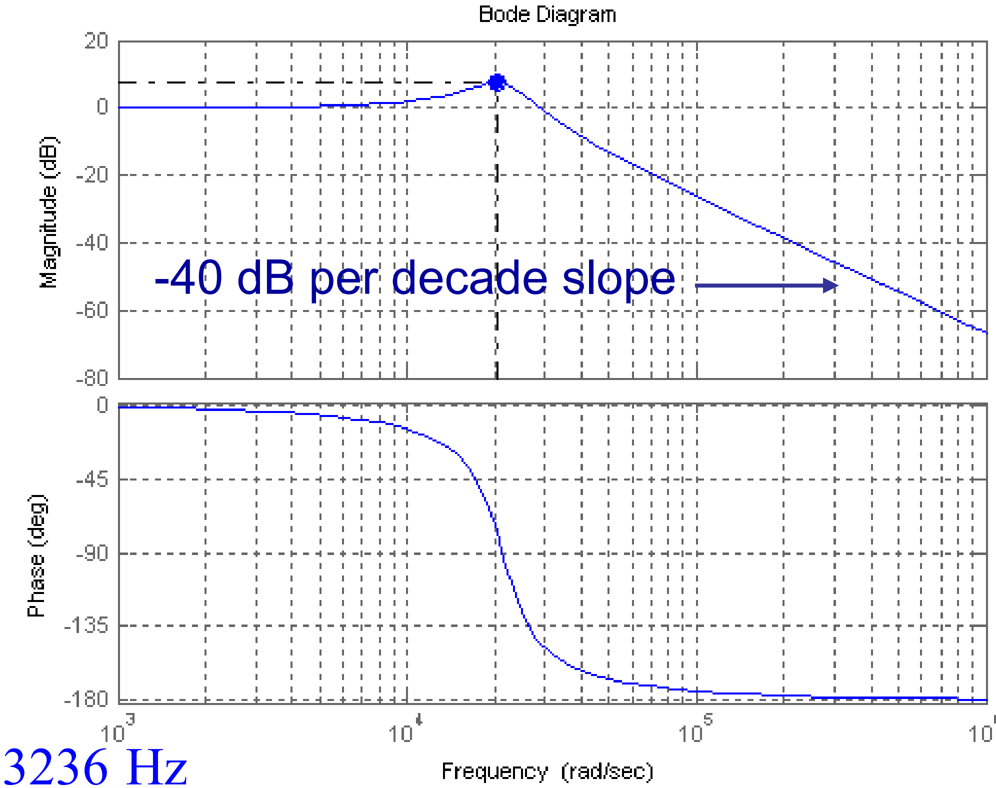


$$t_r \approx \frac{1.8}{\omega_n} = 0.084\ \text{msec}$$

$$t_s \approx \frac{4.6}{\zeta\omega_n} = 1.01\ \text{msec}$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.504$$

Frequency Response



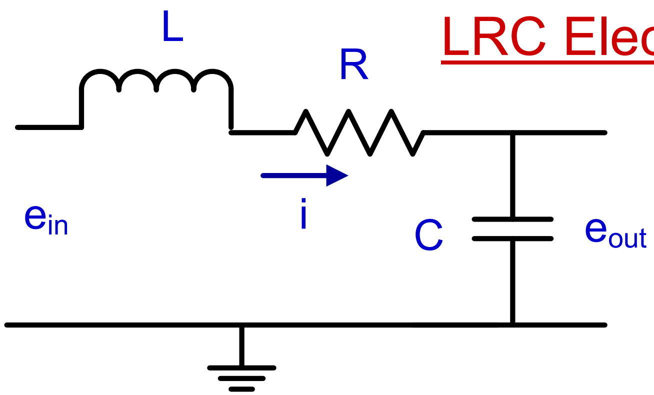
$$\omega_p = \omega_n\sqrt{1-2\zeta^2} = 20330 \text{ rad/sec} = 3236 \text{ Hz}$$

$$\text{peak amplitude ratio} = \frac{K_S}{2\zeta\sqrt{1-\zeta^2}} = 2.403 = 7.61 \text{ dB}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 20830 \text{ rad/sec} = 3315 \text{ Hz}$$

$$BW = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}} = 32057 \text{ rad/sec} = 5120 \text{ Hz}$$

LRC Electrical System



$$e_{in} - e_L - e_R - e_C = 0$$

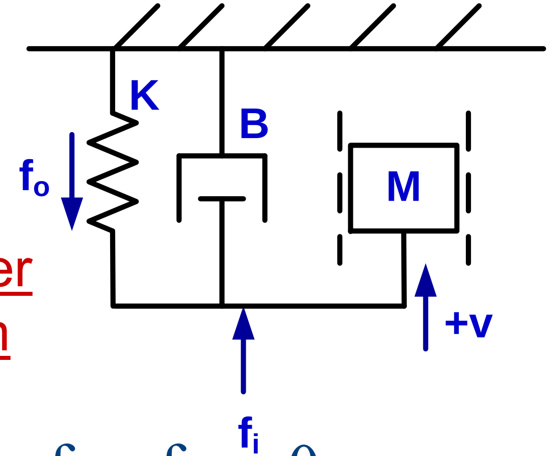
$$e_{in} - L\frac{di}{dt} - Ri - e_{out} = 0$$

$$e_{in} - L\frac{d}{dt}\left(C\frac{de_{out}}{dt}\right) - R\left(C\frac{de_{out}}{dt}\right) - e_{out} = 0$$

$$LC\frac{d^2 e_{out}}{dt^2} + RCdt\frac{de_{out}}{dt} + e_{out} = e_{in}$$

$$\frac{e_{out}}{e_{in}} = \frac{1}{LCD^2 + RCD + 1}$$

Mass-Spring-Damper Mechanical System



$$f_i - f_K - f_B - f_M = 0$$

$$f_i - Kx - Bv - M\dot{v} = 0$$

$$f_i - f_o - B\left(\frac{\dot{f}_o}{K}\right) - M\left(\frac{\ddot{f}_o}{K}\right) = 0$$

$$\frac{M}{K}\ddot{f}_o + \frac{B}{K}\dot{f}_o + f_o = f_i$$

$$\frac{f_o}{f_i} = \frac{1}{\frac{M}{K}D^2 + \frac{B}{K}D + 1}$$

2nd-Order Dynamics Systems

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

$$\frac{1}{\omega_n^2} \frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_0}{dt} + q_0 = K_S q_i$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} = \text{undamped natural frequency}$$

$$\zeta = \frac{a_1}{2\sqrt{a_2 a_0}} = \text{damping ratio}$$

$$K_S = \frac{b_0}{a_0} = \text{steady-state gain}$$

Electrical System

$$LC \frac{d^2 e_{out}}{dt^2} + RC \frac{de_{out}}{dt} + e_{out} = e_{in}$$

$$\omega_n = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad K_S = 1$$

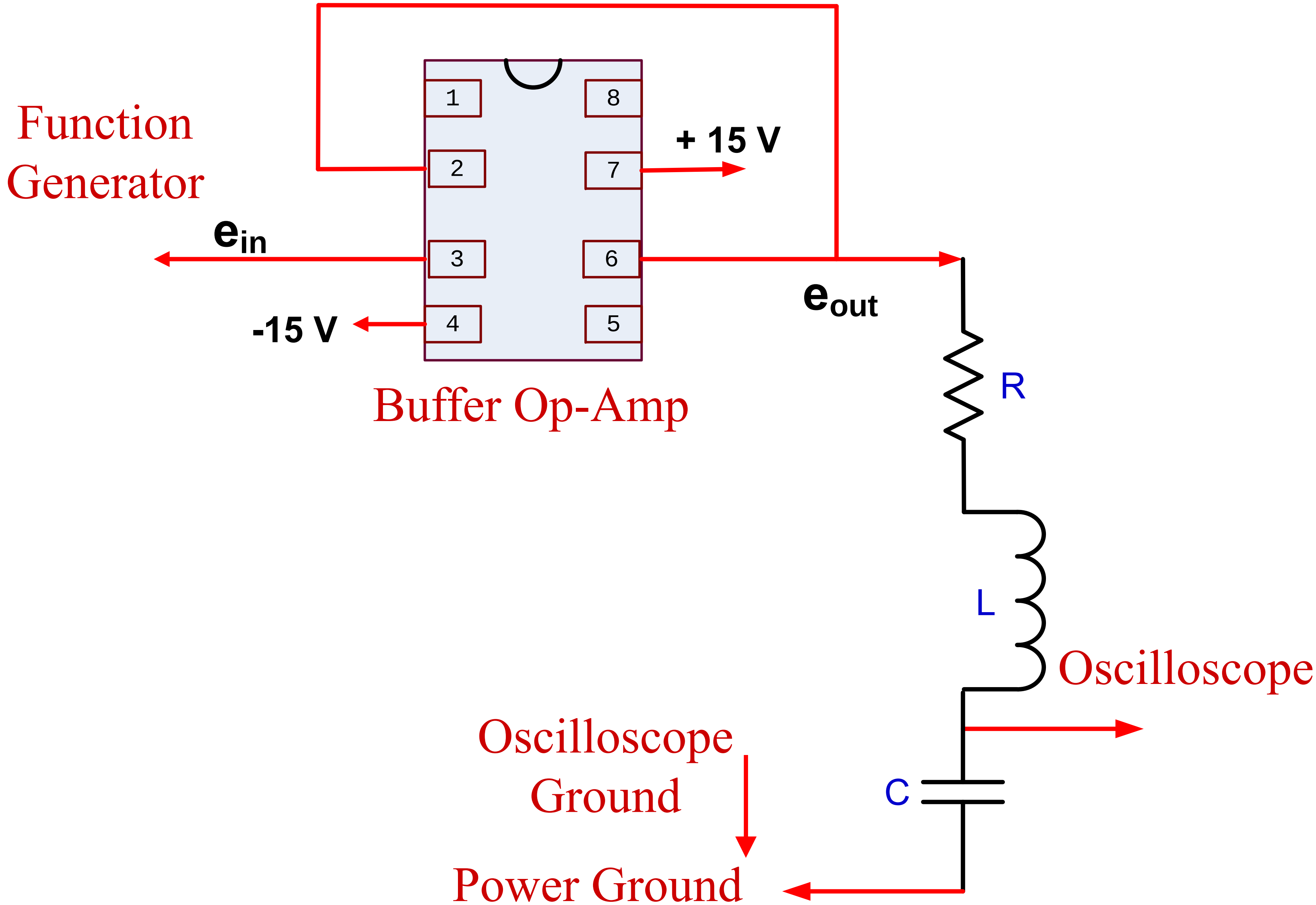
Mechanical System

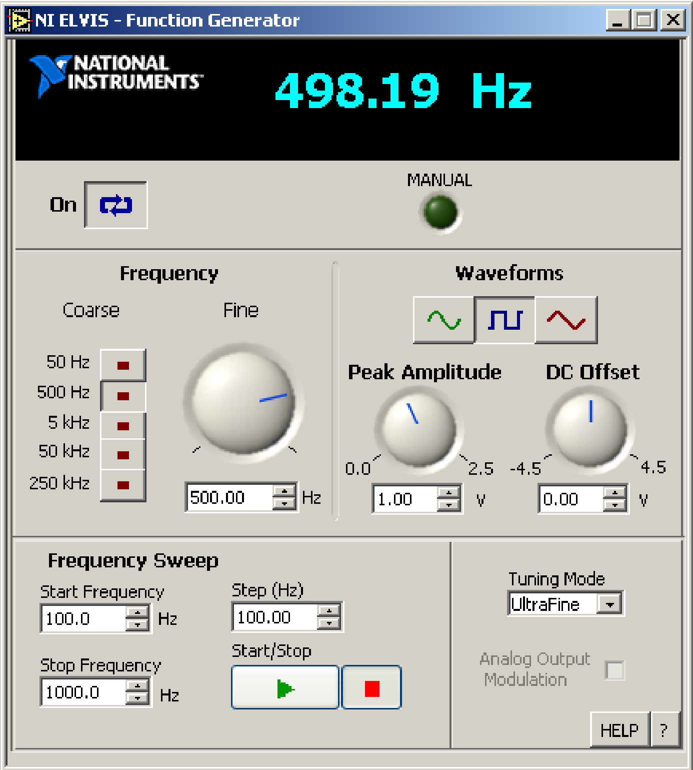
$$\frac{M}{K} \frac{d^2 f_o}{dt^2} + \frac{B}{K} \frac{df_o}{dt} + f_o = f_i$$

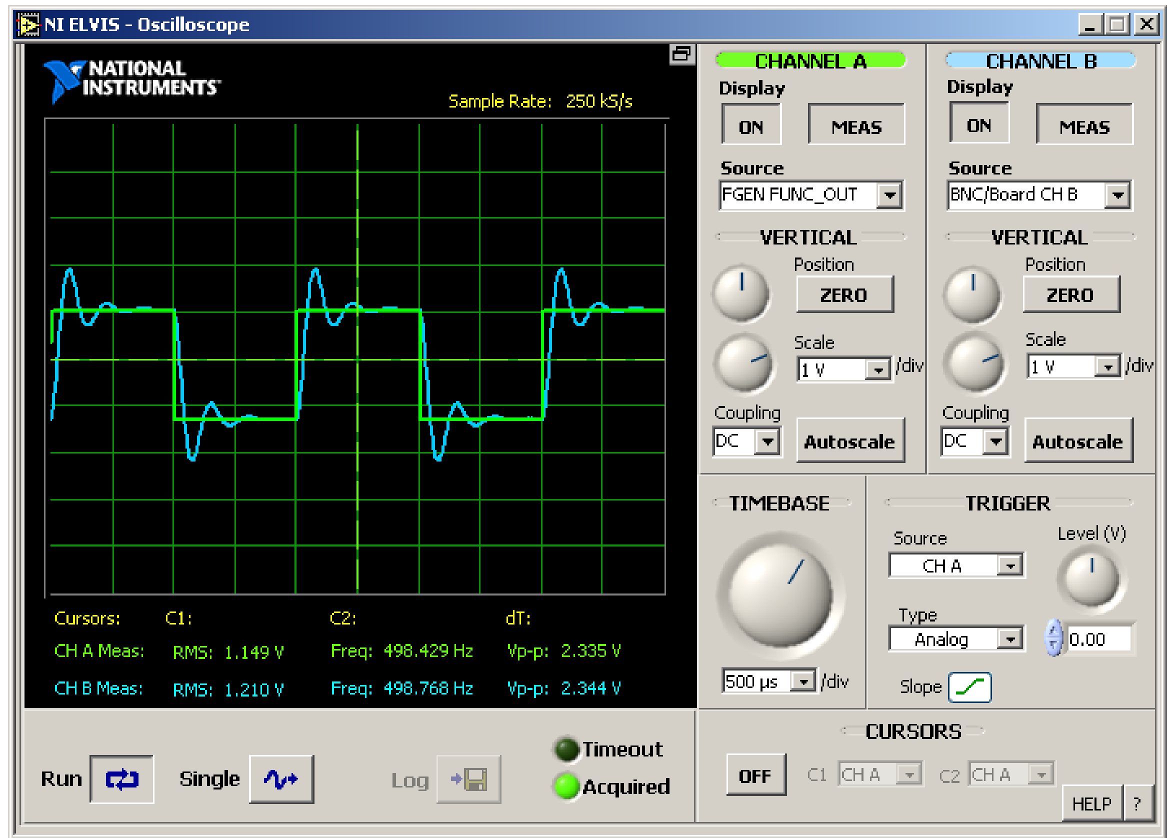
$$\omega_n = \sqrt{\frac{K}{M}} \qquad \zeta = \frac{B}{2}\sqrt{\frac{1}{KM}} \qquad K_S = 1$$

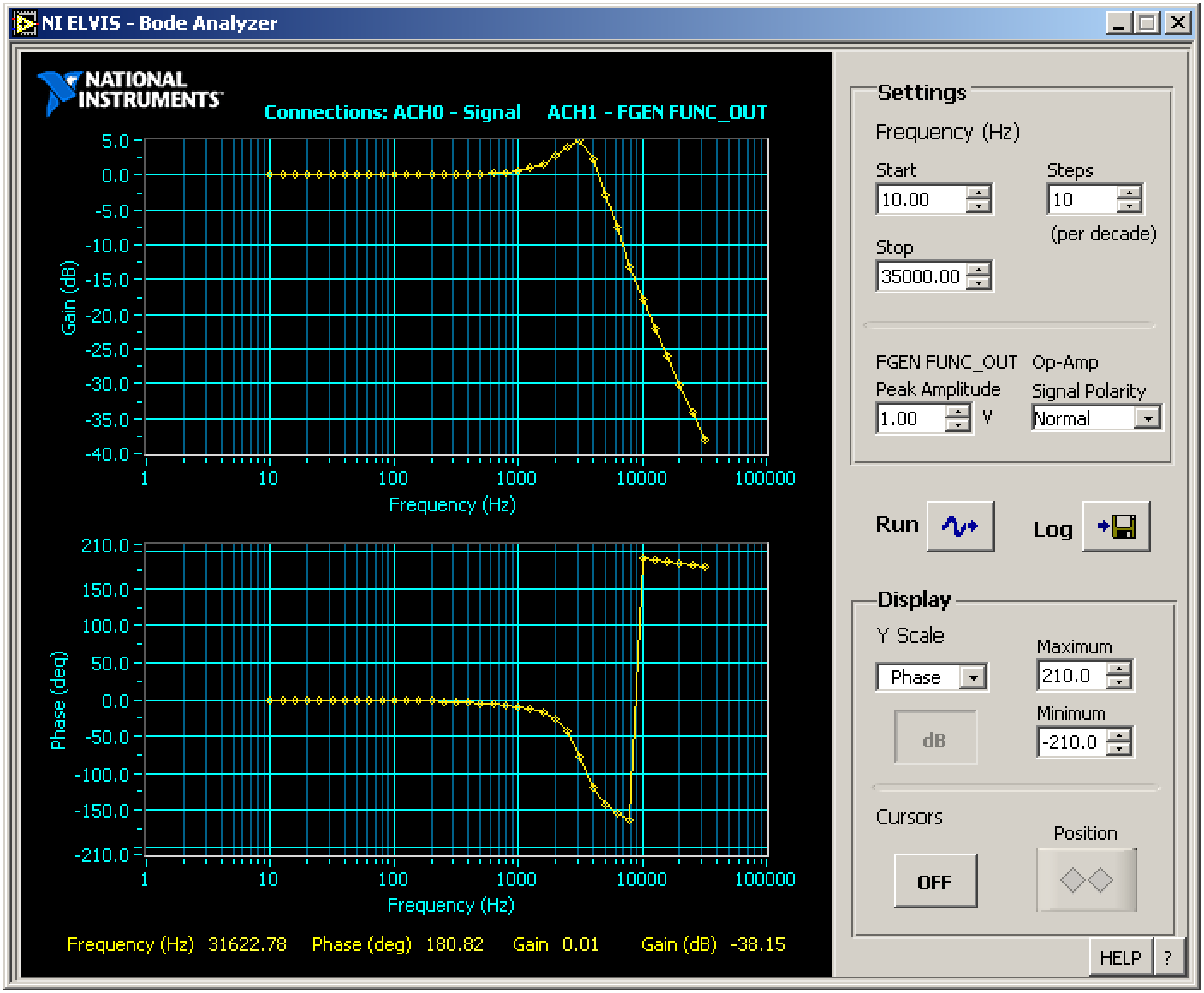
Measurements Using LabVIEW

- LRC Circuit
  - Step Response
  - Frequency Response


Function
Generator
1
2
3
4
5
6
7
8
+ 15 V
e_{in}
-15 V
e_{out}
Buffer Op-Amp
R
L
C
Oscilloscope
Oscilloscope
Ground
Power Ground

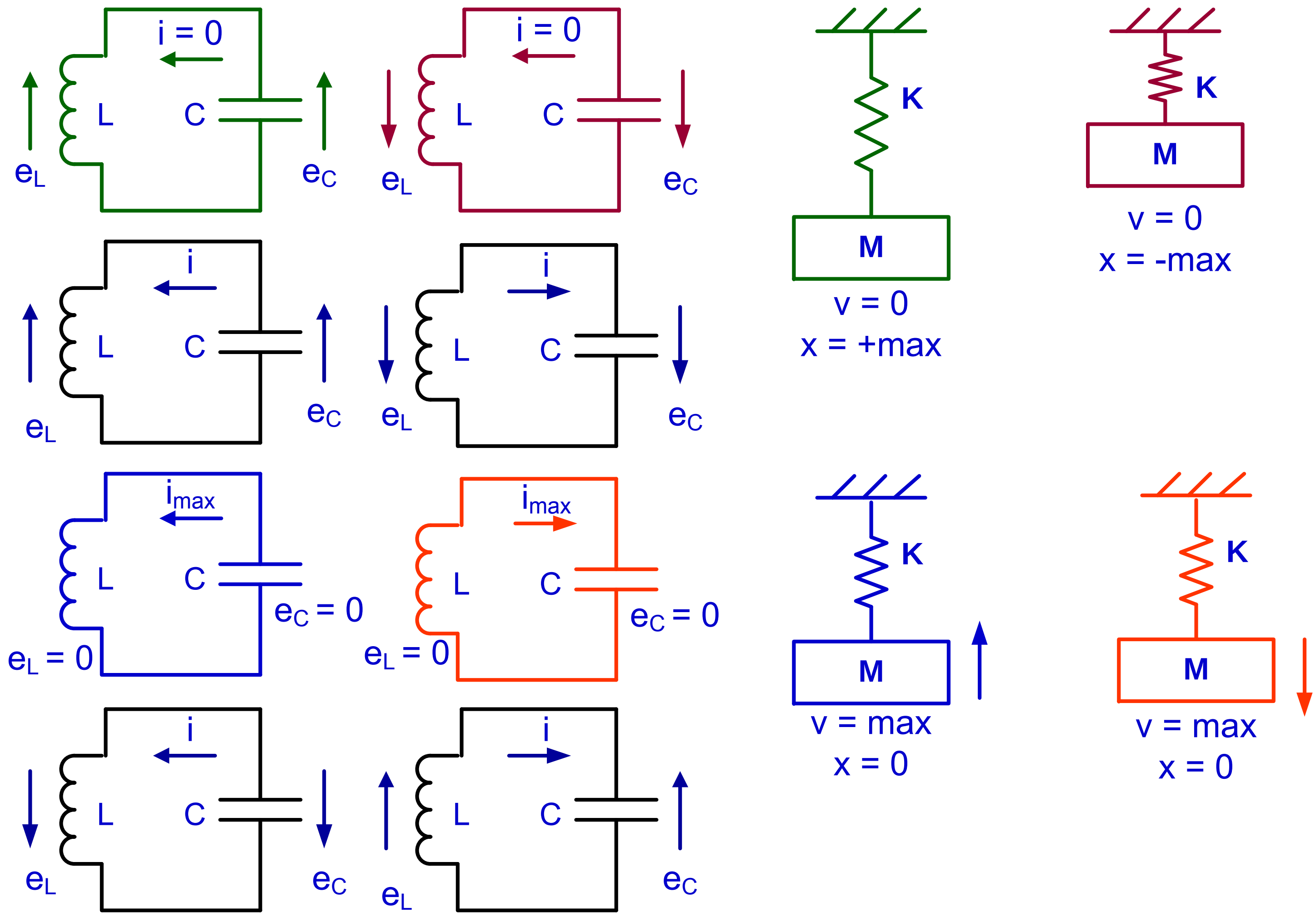

NI ELVIS - Function Generator
NATIONAL INSTRUMENTS
498.19 Hz
On
MANUAL
Frequency
Coarse
Fine
50 Hz
500 Hz
5 kHz
50 kHz
250 kHz
500.00 Hz
Waveforms
Peak Amplitude
DC Offset
0.0
2.5
-4.5
4.5
1.00 V
0.00 V
Frequency Sweep
Start Frequency
100.0 Hz
Step (Hz)
100.00
Stop Frequency
1000.0 Hz
Start/Stop
Tuning Mode
UltraFine
Analog Output Modulation
HELP
?

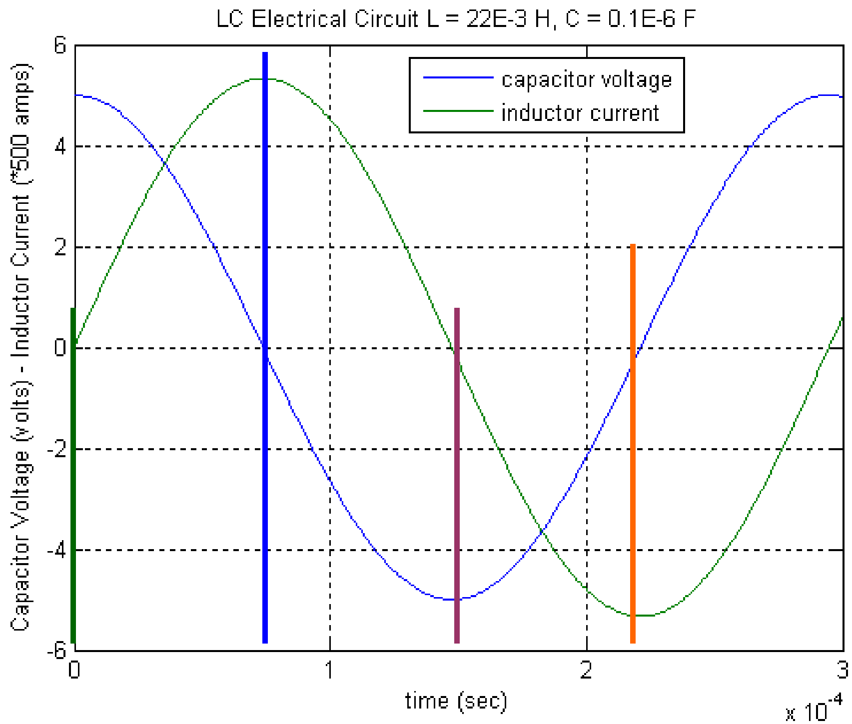

NI ELVIS - Oscilloscope
NATIONAL INSTRUMENTS
Sample Rate: 250 kS/s
Cursors: C1: C2: dT:
CH A Meas: RMS: 1.149 V Freq: 498.429 Hz Vp-p: 2.335 V
CH B Meas: RMS: 1.210 V Freq: 498.768 Hz Vp-p: 2.344 V
CHANNEL A
Display
ON
MEAS
Source
FGEN FUNC_OUT
VERTICAL
Position
ZERO
Scale
1 V /div
Coupling
DC
Autoscale
CHANNEL B
Display
ON
MEAS
Source
BNC/Board CH B
VERTICAL
Position
ZERO
Scale
1 V /div
Coupling
DC
Autoscale
TIMEBASE
500 µs /div
TRIGGER
Source
CH A
Level (V)
Type
Analog
0.00
Slope
CURSORS
OFF
C1 CH A
C2 CH A
Run
Single
Log
Timeout
Acquired
HELP
?

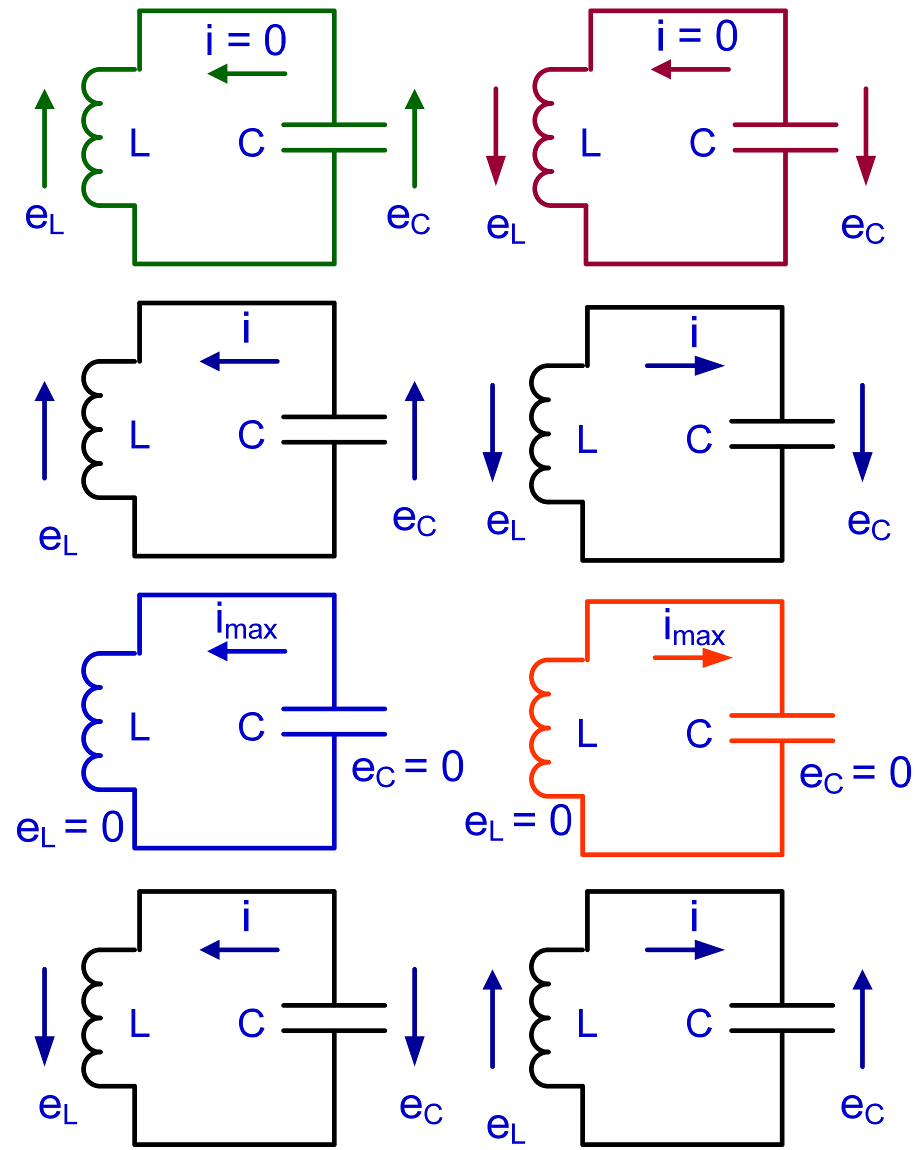

NI ELVIS - Bode Analyzer
NATIONAL INSTRUMENTS
Connections: ACH0 - Signal ACH1 - FGEN FUNC_OUT
Gain (dB)
Frequency (Hz)
Phase (deg)
Frequency (Hz)
Frequency (Hz) 31622.78 Phase (deg) 180.82 Gain 0.01 Gain (dB) -38.15
Settings
Frequency (Hz)
Start
10.00
Steps
10
(per decade)
Stop
35000.00
FGEN FUNC_OUT Peak Amplitude
1.00
V
Op-Amp Signal Polarity
Normal
Run
Log
Display
Y Scale
Phase
dB
Maximum
210.0
Minimum
-210.0
Cursors
OFF
Position
HELP
?

LC Circuit Resonance

Inductor-Capacitor (LC) ↔ Mass-Spring (MK) Oscillations







$$\text{KVL}: \quad e_C + e_L = 0$$

$$i = C\frac{de_C}{dt}$$

$$e_L = L\frac{di}{dt} = L\frac{d}{dt}\left(C\frac{de_C}{dt}\right) = LC\frac{d^2e_C}{dt^2}$$

$$LC\frac{d^2e_C}{dt^2} + e_C = 0 \qquad \omega_n = \frac{1}{\sqrt{LC}}$$