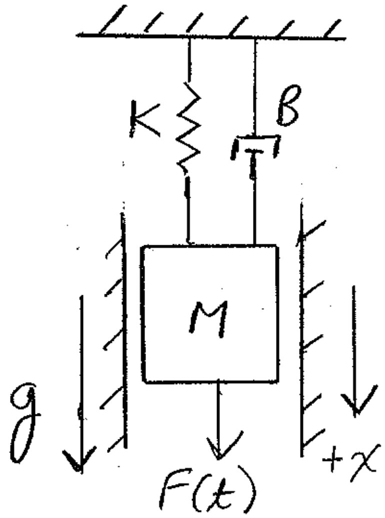
$\boxed{1.}$ Spring-Mass-Damper System

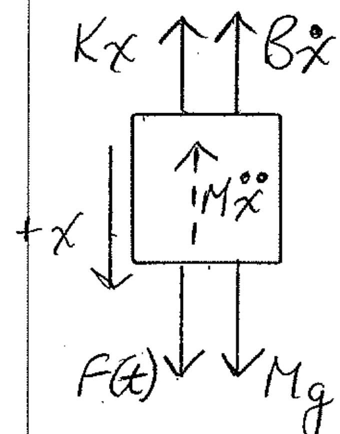


Physical Model

- One-degree-of-freedom system
- Pure and ideal spring, mass, and damper
- Rigid support
- Frictionless side walls
- Gravity acts vertically down

Simplifying Assumptions

FBD



x is measured from unstretched spring length

$M\ddot{x}$ = inertia force (fictitious)

Mathematical Model

$\sum F_x = 0 \quad \downarrow +x$

$$F(t) + Mg - M\ddot{x} - B\dot{x} - Kx = 0$$

$$M\ddot{x} + B\dot{x} + Kx = F(t) + Mg$$

$$x_{total} \equiv x_{static} + x_{dynamic} = x_s + x_d$$

In static equilibrium

$$\dot{x} = \ddot{x} = 0 \qquad x = x_s$$

$$F(t) = 0$$

$$Kx_s = Mg \Rightarrow x_s = \frac{Mg}{K}$$

x_s = static stretch of spring

$$M(\ddot{x}_s + \ddot{x}_d) + B(\dot{x}_s + \dot{x}_d) + K(x_s + x_d) = F(t) + Mg$$

$$\ddot{x}_s = \dot{x}_s = 0$$

$$M\ddot{x}_d + B\dot{x}_d + Kx_s + Kx_d = F(t) + Mg$$

<u>But</u> $Kx_s = Mg$

$$M\ddot{x} + B\dot{x} + Kx = F(t) \implies$$

$x = x_d$, i.e., x is measured from the static equilibrium position.

$D = \frac{d}{dt}$ = differential operator

$$MD^2x + BDx + Kx = F(t)$$

$$(MD^2 + BD + K)x = F(t)$$

$$\frac{x}{F} = \frac{1}{MD^2 + BD + K}$$

$$\frac{x}{F} = \frac{1/M}{D^2 + \frac{B}{M}D + \frac{K}{M}}$$

$$\frac{Kx}{F} = \frac{K/M}{D^2 + \frac{B}{M}D + \frac{K}{M}} = \frac{\omega_n^2}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

$$\left\{ \begin{array}{l} \omega_n^2 = K/M \\ 2\zeta\omega_n = \frac{B}{M} \end{array} \right.$$

$\omega_n = \sqrt{\frac{K}{M}}$

$\zeta = \frac{B}{2}\sqrt{\frac{1}{KM}}$

<u>Transfer Function</u>

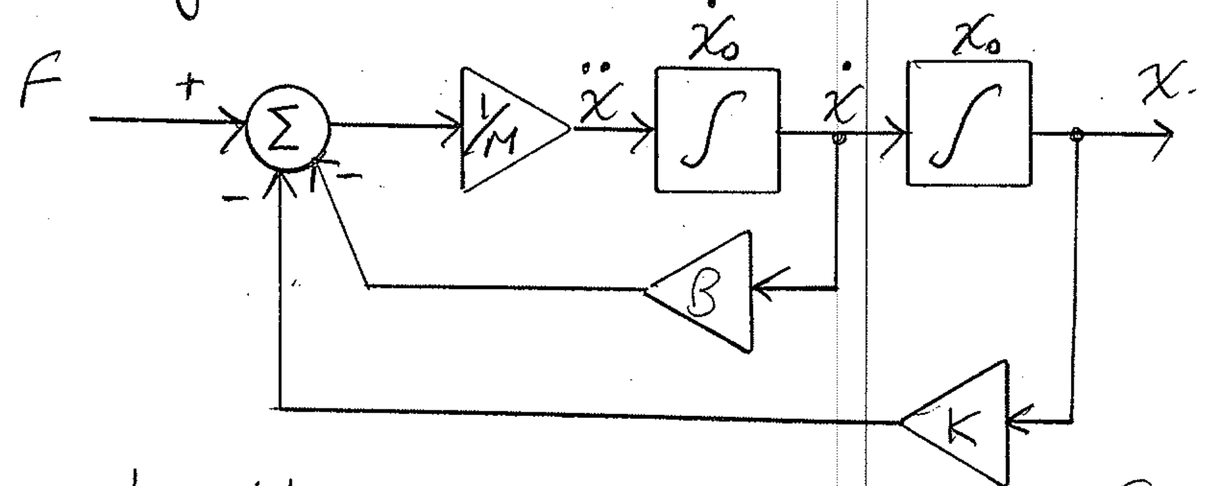
Energy Characteristics:

Spring stores energy → $PE = \frac{1}{2} K x^2$

Mass stores energy → $KE = \frac{1}{2} M v^2$

Damper dissipates energy → $\int \left(B \frac{dx}{dt}\right) dx$

Block Diagram



$$\ddot{x} = \frac{1}{M}\left[-B\dot{x} - Kx + F\right]$$

State Variables

$q_1 = x \qquad q_2 = v$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

A Matrix   B Matrix

$$\begin{cases} \dot{q}_1 = \dot{x} = v = q_2 \\ \dot{q}_2 = \dot{v} = \frac{1}{M}\left[-Bv - Kx + F\right] \\ \quad = \frac{1}{M}\left[-Bq_2 - Kq_1 + F\right] \end{cases}$$

$u = F(t)$

Electrical-Mechanical Analogy

$f \leftrightarrow e$

$v \leftrightarrow i$

$B \leftrightarrow R$

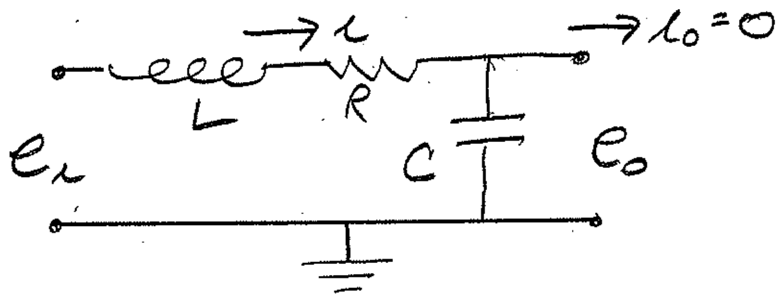
$K \leftrightarrow 1/C$

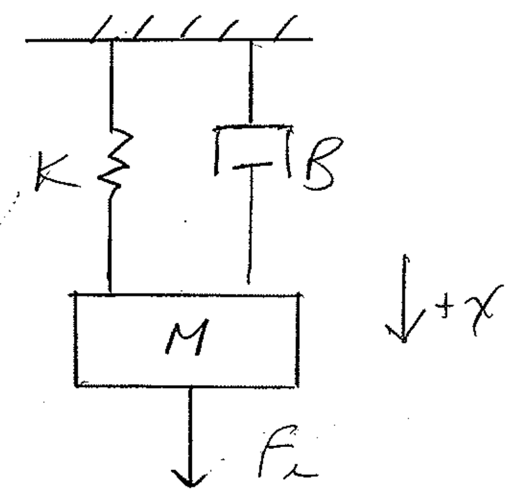
$M \leftrightarrow L$

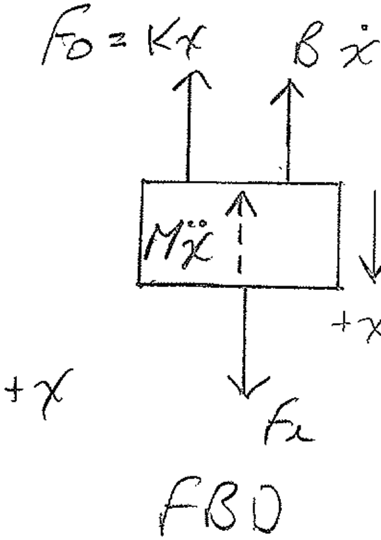
$e = Ri \qquad f = Bv$

$e = L\dot{i} \qquad f = M\dot{v}$

$e = \frac{1}{C}\int i\,dt \qquad f = K\int v\,dt$







$$\frac{e_o}{e_i} = \frac{1/CD}{LD + R + 1/CD}$$

$$= \frac{1}{LCD^2 + RCD + 1}$$

Voltage Divider

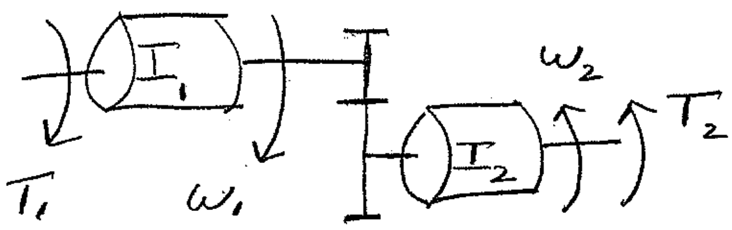
Compare

$M\ddot{x} + B\dot{x} + Kx = F_i$

$(MD^2 + BD + K)x = F_i$

$$\frac{Kx}{F_i} = \frac{F_D}{F_i} = \frac{1}{\frac{M}{K}D^2 + \frac{B}{K}D + 1} \Leftarrow \frac{x}{F_i} = \frac{1}{MD^2 + BD + K}$$

2.



$$N = \frac{\omega_1}{\omega_2}$$

(a) <u>In terms of ω_1</u>

$\omega_2 \rightarrow \omega_1$

$$\omega_2 = \frac{1}{N}\omega_1$$

$$\left(I_1 + \frac{I_2}{N^2}\right)\dot{\omega}_1 = T_1 + \frac{T_2}{N}$$

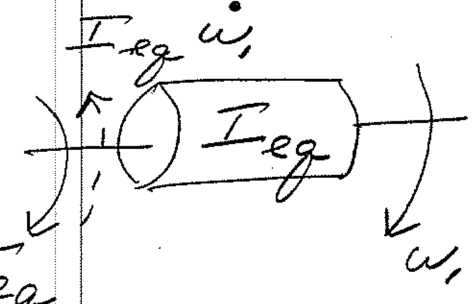
(b) <u>In term of ω_2</u>

$\omega_1 \rightarrow \omega_2$

$$\omega_1 = N\omega_2$$

$$\left(I_2 + N^2 I_1\right)\dot{\omega}_2 = T_2 + N T_1$$

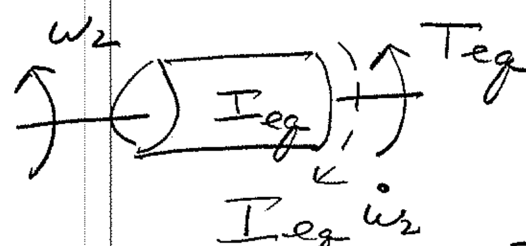
- One-Degree-of-freedom System
- Equation of Motion can be written in terms of ω_1 or ω_2.



$$I_{eq}\dot{\omega}_1 = T_{eq}$$

$$I_{eq} = I_1 + \frac{I_2}{N^2}$$

$$T_{eq} = T_1 + \frac{T_2}{N}$$



$$I_{eq}\dot{\omega}_2 = T_{eq}$$

$$I_{eq} = I_2 + N^2 I_1$$

$$T_{eq} = T_2 + N T_1$$

(c) Energy Approach $\Rightarrow$ Equivalent Inertia

$$KE_{system} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} I_{eq} \omega_1^2$$

$$\frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \left(\frac{\omega_1}{N}\right)^2 = \frac{1}{2}\left(I_1 + \frac{I_2}{N^2}\right)\omega_1^2$$

$$I_{eq} = I_1 + \frac{I_2}{N^2} \quad \text{referred to } \omega_1$$

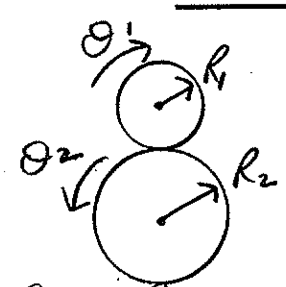
Similarly,

$$KE_{system} = \frac{1}{2} I_1 (N^2 \omega_2^2) + \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} I_{eq} \omega_2^2$$

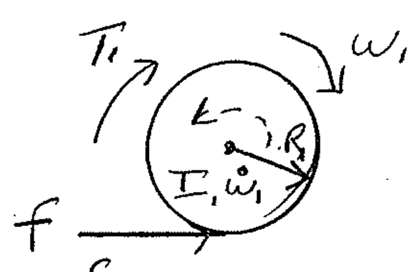
$$\frac{1}{2}\left(I_1 N^2 + I_2\right)\omega_2^2 = \frac{1}{2} I_{eq} \omega_2^2$$

$$I_{eq} = I_2 + I_1 N^2 \quad \text{referred to } \omega_2$$

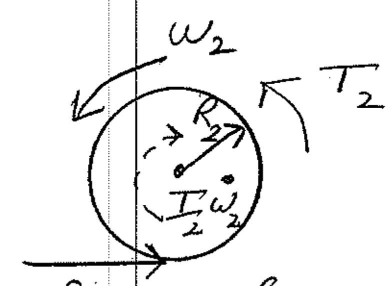
(d) FBD Approach



$R_1\theta_1 = R_2\theta_2$
gear train



gear force ON 1 from 2



gear force ON 2 from 1

$$N = \frac{\omega_1}{\omega_2} = \frac{\theta_1}{\theta_2}$$

$$R_1\theta_1 = R_2\theta_2$$

$$\frac{\theta_1}{\theta_2} = \frac{R_2}{R_1} = N$$

$$\sum M = 0$$

$$T_1 - I_1\dot{\omega}_1 - fR_1 = 0 \Rightarrow I_1\dot{\omega}_1 + fR_1 = T_1 \qquad [1]$$

$$T_2 - I_2\dot{\omega}_2 + fR_2 = 0 \Rightarrow I_2\dot{\omega}_2 - fR_2 = T_2 \qquad [2]$$

$$[1] \rightarrow \quad f = \frac{1}{R_1}\left(T_1 - I_1\dot{\omega}_1\right)$$

$$[2] \rightarrow \quad I_2\dot{\omega}_2 - \left[\frac{1}{R_1}\left(T_1 - I_1\dot{\omega}_1\right)\right]R_2 = T_2$$

$$I_2\dot{\omega}_2 - \frac{R_2}{R_1}\left(T_1 - I_1\dot{\omega}_1\right) = T_2$$

$$I_2\dot{\omega}_2 - N\left(T_1 - I_1 N\dot{\omega}_2\right) = T_2$$

$$\left(I_2 + I_1 N^2\right)\dot{\omega}_2 - NT_1 = T_2$$

$$\left(I_2 + I_1 N^2\right)\dot{\omega}_2 = T_2 + NT_1$$

same as previously derived

Equation of Motion in terms of ω_2:

$$\left(I_2 + N^2 I_1\right)\dot{\omega}_2 = T_2 + N T_1$$

$$\dot{\omega}_2 = \frac{T_2 + N T_1}{I_2 + N^2 I_1}$$

$$\frac{d\dot{\omega}_2}{dN} = 0 = \frac{\left(I_2 + N^2 I_1\right)\left(T_1\right) - \left(T_2 + N T_1\right)\left(2 N I_1\right)}{\left(I_2 + N^2 I_1\right)^2}$$

$$\left(I_2 + N^2 I_1\right)\left(T_1\right) - \left(T_2 + N T_1\right)\left(2 N I_1\right) = 0$$

$$T_1 I_2 + N^2 I_1 T_1 - 2 N I_1 T_2 - 2 N^2 T_1 I_1 = 0$$

$$-N^2 T_1 I_1 - 2 N I_1 T_2 + T_1 I_2 = 0$$

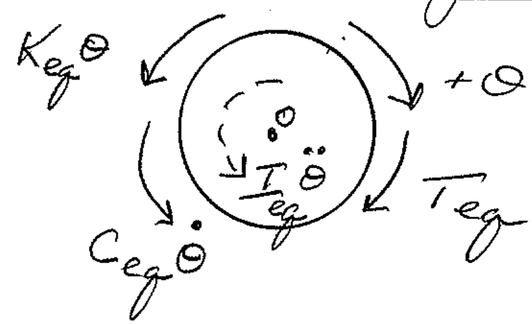
$$N^2 T_1 I_1 + 2 N I_1 T_2 - T_1 I_2 = 0$$

For $T_2 = 0 \Rightarrow \quad N^2 T_1 I_1 = T_1 I_2$

$$N^2 I_1 = I_2 \qquad N = \sqrt{I_2 / I_1}$$

for max $\alpha_2 = \dot{\omega}_2$, the reflected inertia $N^2 I_1$ should $= I_2$.

3. **Rotational Equivalent**



$$\sum M_o = 0$$

$$I_{eq}\ddot{\theta} + C_{eq}\dot{\theta} + K_{eq}\theta = T_{eq}$$

Kinematic Relationship

$$x = R\theta$$

$x \rightarrow \theta$

$x = R\theta$

$$\left\{ \begin{array}{l} I_{eq} = (I_m + I_r + I_p + m_r R^2) \\ C_{eq} = C R^2 \\ K_{eq} = K R^2 \end{array} \right.$$

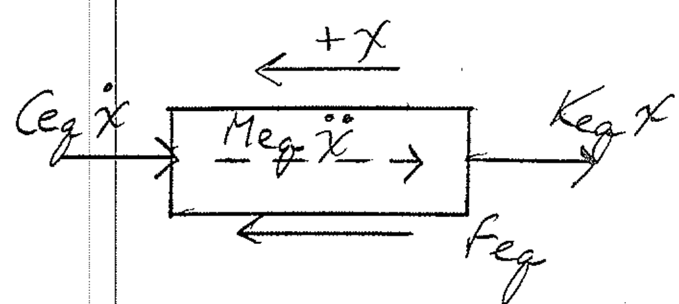
$$T_{eq} = T$$

Equation of Motion

$$\left[I_m + I_r + I_p + m_r R^2\right]\ddot{\theta} + CR^2\dot{\theta} + KR^2\theta = T$$

Translational Equivalent

$\sum f_x = 0 \quad M_{eq}\ddot{x} + C_{eq}\dot{x} + K_{eq}x = f_{eq}$



$\theta \rightarrow x$

$\theta = \frac{1}{R}x$

$M_{eq} = \left(m_r + \frac{I}{R^2}\right)$ where $I = I_m + I_s + I_p$

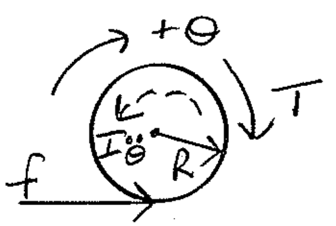
$C_{eq} = C$

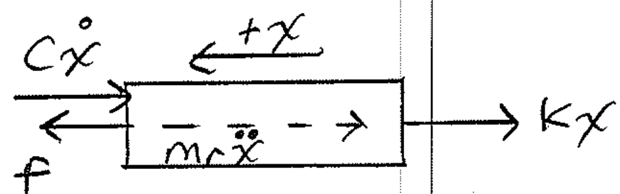
$K_{eq} = K$ $\quad f_{eq} = \frac{1}{R} T$

Equation of Motion

$$\left[m_r + \frac{I}{R^2}\right]\ddot{x} + C\dot{x} + Kx = \frac{T}{R}$$

<u>Alternate Approach</u>





$I = I_m + I_s + I_p$

$\sum M_o = 0$

$-I\ddot{\theta} + T - fR = 0$

$I\ddot{\theta} + fR = T$ [1]

$\sum f_x = 0$

$-c\dot{x} + f - Kx - m_r\ddot{x} = 0$

$m_r\ddot{x} + c\dot{x} + Kx = f$ [2]

Kinematic Relations

$$\begin{cases} x = R\theta \\ \dot{x} = R\dot{\theta} \\ \ddot{x} = R\ddot{\theta} \end{cases}$$

Solve for f in [2] and substitute into [1]:

$$I\ddot{\theta} + R\left[m_r\ddot{x} + c\dot{x} + kx\right] = T$$

$$I\ddot{\theta} + R\left[m_r R\ddot{\theta} + cR\dot{\theta} + kR\theta\right] = T$$

$$\left[I + m_r R^2\right]\ddot{\theta} + cR^2\dot{\theta} + kR^2\theta = T \quad [3]$$

Solve for f in [1] and substitute into [2]:

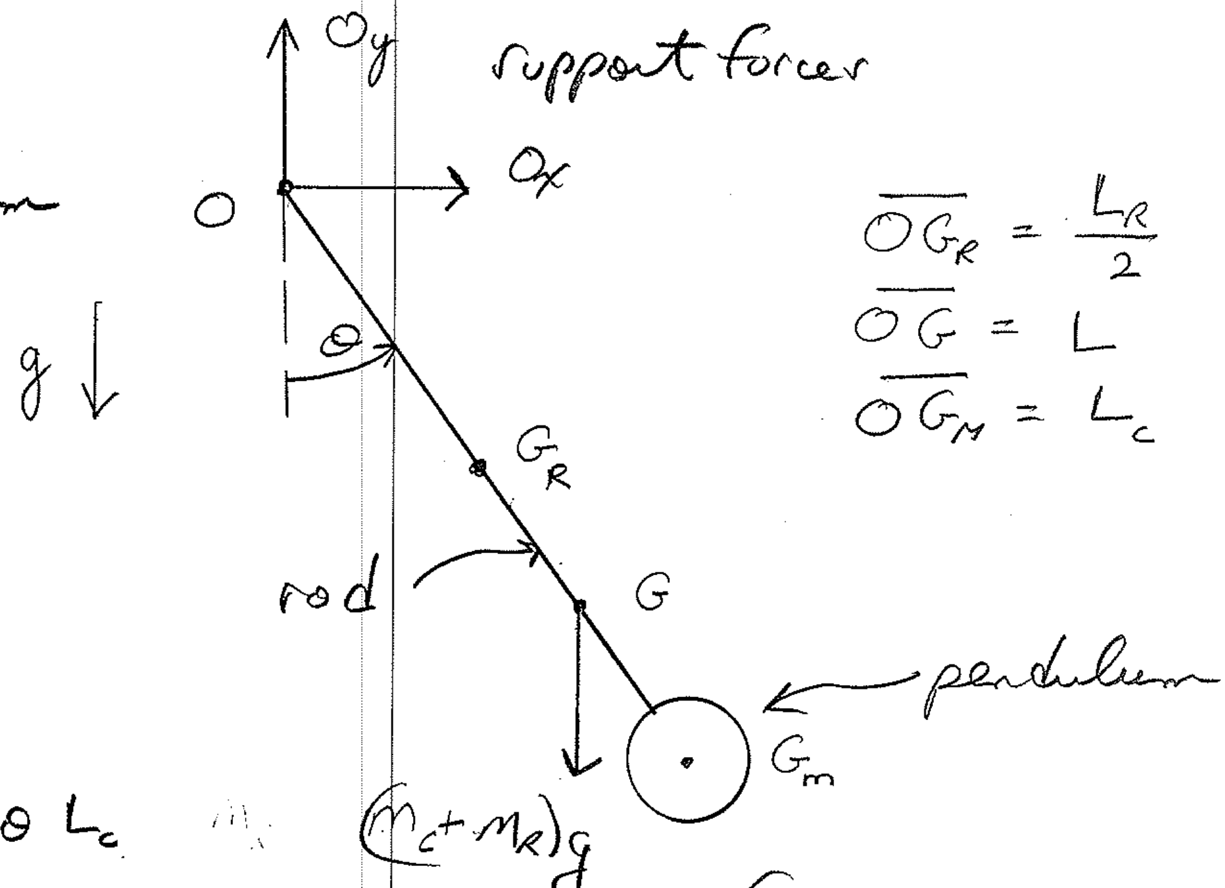
$$m_r\ddot{x} + c\dot{x} + kx = \frac{T - I\ddot{\theta}}{R}$$

$$m_r\ddot{x} + c\dot{x} + kx = \frac{T}{R} - \frac{I}{R}\frac{\ddot{x}}{R}$$

$$\left[m_r + \frac{I}{R^2}\right]\ddot{x} + c\dot{x} + kx = \frac{T}{R} \quad [4]$$

4. FBD

m_c = mass of pendulum
m_R = mass of rod



$\overline{OG_R} = \frac{L_R}{2}$
$\overline{OG} = L$
$\overline{OG_M} = L_c$

G = center of mass for system
G_m = center of mass of pendulum
G_R = center of mass of rod

Relationship among L_R, L, and L_c:

$$m_R g \sin\theta \frac{L_R}{2} + m_c g \sin\theta L_c = (m_R+m_c) g \sin\theta L$$

$$m_R \frac{L_R}{2} + m_c L_c = (m_R+m_c) L$$

$$L = \frac{m_R \frac{L_R}{2} + m_c L_c}{m_R+m_c}$$

Summation of Moments about pt. O must be same.

Fixed-Axis Rotation

Equation of Motion: $\sum M_o = \bar{I}_o \ddot{\theta}$

$$\sum M_o = (m_c + m_R) g L \sin\theta$$

$$\bar{I}_o = \bar{I}_{Rod} + m_R\left(\frac{L_R}{2}\right)^2 + \bar{I}_{pendulum} + m_c L_c^2$$

Parallel Axis Theorem

$$I_o = \bar{I}_{RG} + m_R\left(\frac{L_R}{2}\right)^2 + 0 + m_c L_c^2$$

pendulum is modeled as a point mass.

Therefore

$$\left[\bar{I}_{RG} + m_R\left(\frac{L_R}{2}\right)^2 + m_c L_c^2\right]\ddot{\theta} + (m_c + m_R) g L \sin\theta = 0$$

Nonlinear equation $\rightarrow$ $\sin\theta$

$\sin\theta \approx \theta$ for small angles

$$\sin\theta \cong (\sin\theta)\Big|_{\theta=0^\circ} + \left(\frac{d\sin\theta}{d\theta}\right)\Big|_{\theta=0^\circ} \theta + \ldots = 0 + (\cos 0^\circ)\theta = \theta$$

Taylor Series Expansion about $\theta = 0^\circ$.

Linear Equation of Motion

$$\left[I_{RG} + m_R\left(\frac{L_R}{2}\right)^2 + m_c L_c^2\right]\ddot{\theta} + (m_c + m_R)\, g\, L\theta = 0$$

- Solve analytically
- Insight into dynamic behavior
- Facilitates control design

$\Big\}$ Why we linearize.

Also most systems operate close to some operating point and we are interested in behavior near that operating point.

If $m_R \ll m_c$, then $I_{RG} = 0$ and $m_R = 0$

$m = m_c$ and $L = L_c$

Equation of Motion becomes:

$$m_c L^2 \ddot{\theta} + m_c g L \theta = 0$$

$$L\ddot{\theta} + g\theta = 0$$

$$\ddot{\theta} + g/L\, \theta = 0$$

note that equation is independent of m_c →

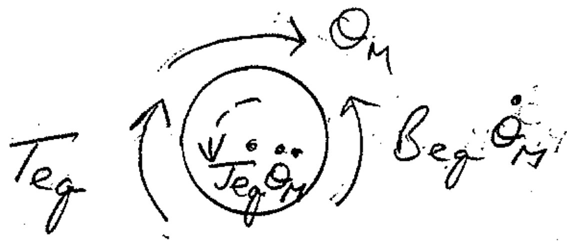
5. Belt-Driven System — Rigid Belt

$R_M \theta_M = R_L \theta_L$

$N = \dfrac{R_L}{R_M}$ $\qquad \theta_L = \left(\dfrac{R_M}{R_L}\right) \theta_M$

Kinematic Relationship

Equivalent System θ_M

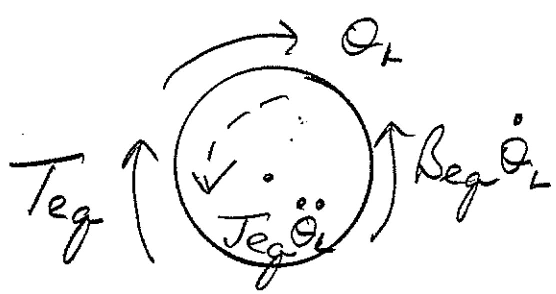


$\theta_L = \dfrac{1}{N}\theta_M$

$$\left.\begin{aligned} T_{eq} &= T_M - \frac{T_L}{N} \\ B_{eq} &= B_M + B_L\left(\frac{1}{N^2}\right) \\ J_{eq} &= J_M + J_L\left(\frac{1}{N^2}\right) \end{aligned}\right\}$$

$$J_{eq}\ddot{\theta}_M + B_{eq}\dot{\theta}_M = T_{eq}$$

Equivalent System θ_L



$\theta_M = N\theta_L$

$$\left.\begin{aligned} T_{eq} &= -T_L + T_M N \\ B_{eq} &= B_L + B_M(N^2) \\ J_{eq} &= J_L + J_M(N^2) \end{aligned}\right\}$$

$$J_{eq}\ddot{\theta}_L + B_{eq}\dot{\theta}_L = T_{eq}$$

Belt-Driven System – Rigid Belt

$$\theta_M = N\,\theta_L \qquad N = \frac{R_L}{R_M} \begin{array}{l} \leftarrow \text{radius of load pulley} \\ \leftarrow \text{radius of motor pulley} \end{array}$$

Same direction of rotation

Gear-Driven System – Ideal Gear Train

$$\omega_M = N\,\omega_L \qquad N = \frac{R_L}{R_M} \begin{array}{l} \leftarrow \text{radius of load gear} \\ \leftarrow \text{radius of motor gear} \end{array}$$

Opposite direction of rotation

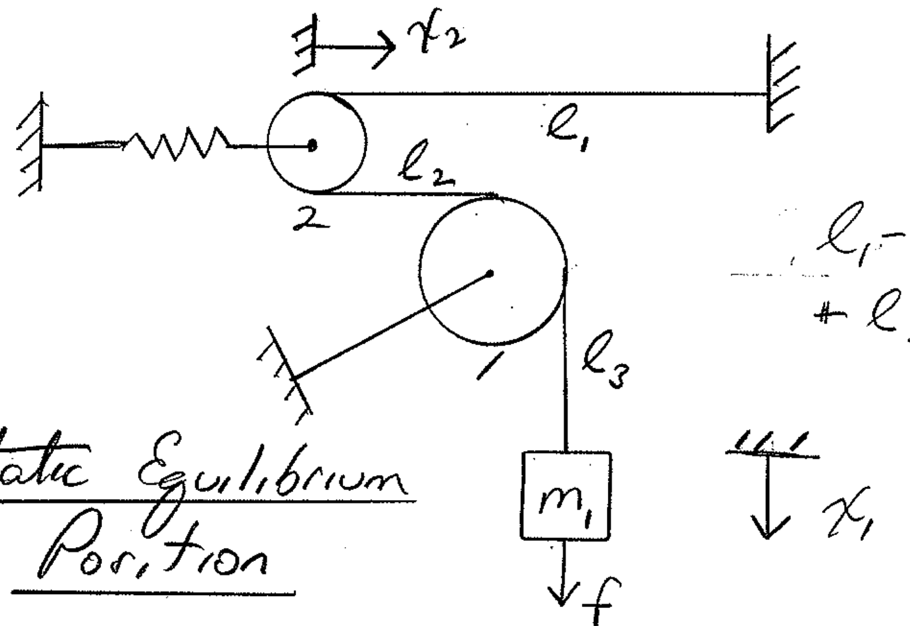
$$\left.\begin{array}{l}\text{Equations of Motion are the same.} \\ \text{Transfer Functions are the same.}\end{array}\right\} \ast$$

6. Pulley – Cable System

Assumptions:

- pure and ideal spring
- massless, inextensible cable
- frictionless pulleys
- no slip of cable on pulleys
- pulley 2 mass is negligible

Kinematic Constraint: length of cable is constant.



$$\ell_1 - x_2 + \ell_2 - x_2 + \ell_3 + x_1 = C_1$$

$$x_1 - 2x_2 = C_2$$

x_1 and x_2 are measured from the system static equilibrium position.

Static Equilibrium Position

$x_1 = x_2 = 0$

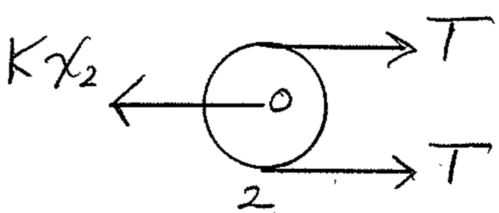
$$\Delta x_1 - 2\Delta x_2 = 0$$

$$\Delta x_1 = 2\Delta x_2$$

First, assume that pulley 1 is also massless.
If that is assumed, then the tension T in the cable is the same throughout.

FBO

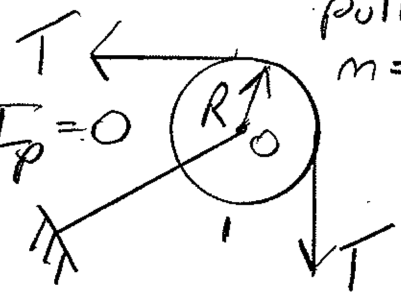
pulley 2 massless
$m=0,\ I=0$



$$2T = Kx_2$$
$$T = \frac{K}{2}x_2$$

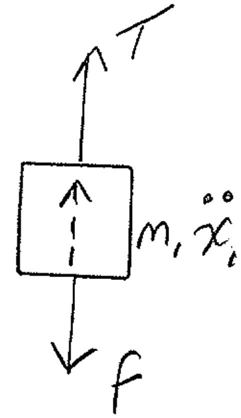
$\sum M_o = 0 \quad \sum F = 0$

pulley 1
$m=0,\ I=0$



$I_p = 0$

$\sum M_o = 0$



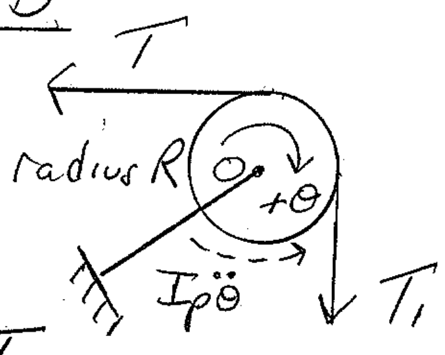
$$\sum F = 0$$
$$-m_1\ddot{x}_1 - T + f = 0$$
$$m_1\ddot{x}_1 + T = f$$
$$m_1\ddot{x}_1 + \frac{K}{2}x_2 = f$$
$$m_1\ddot{x}_1 + \frac{K}{2}\left(\frac{x_1}{2}\right) = f$$

Equation of Motion $\longrightarrow$ $m_1\ddot{x}_1 + \frac{K}{4}x_1 = f$

$m_1 g$ does not appear in FBO since we are measuring x_1 and x_2 from static equil. position.

Now assume that pulley 2 has a mass moment of inertia I_p and a mass m_p.

FBD



$\Sigma M_O = 0$

$T_1 R - TR - I_p \ddot{\theta} = 0$

$I_p \ddot{\theta} = (T_1 - T) R$

$T_1 \neq T$

$\left.\begin{array}{l} T = \frac{K}{2} x_2 \\ \Delta x_1 = 2 \Delta x_2 \end{array}\right\}$

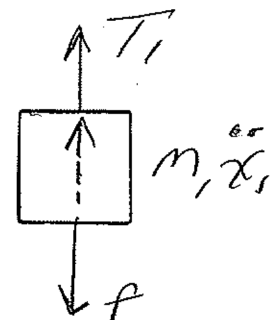
stays the same.

Kinematic Relationship :

No slip

$R\theta = x_1$

FBD



$m_1 \ddot{x}_1 + T_1 = f$

$I_p \frac{\ddot{x}_1}{R} = \left[T_1 - \frac{K}{2}\left(\frac{x_1}{2}\right) \right] R$

$I_p \ddot{x}_1 = T_1 R^2 - \frac{K}{4} x_1 R^2$

$T_1 = \frac{1}{R^2}\left[I_p \ddot{x}_1 + \frac{K}{4} x_1 R^2 \right]$

substitute

Substitution:

$$m_1 \ddot{x}_1 + T_1 = f$$

$$m_1 \ddot{x}_1 + \frac{1}{R^2}\left[I_p \ddot{x}_1 + \frac{K}{4} x_1 R^2\right] = f$$

$$\left[m_1 + \frac{I_p}{R^2}\right] \ddot{x}_1 + \frac{K}{4} x_1 = f$$

Equation of Motion

Compare Equation:

pulley 1 massless: $m_1 \ddot{x}_1 + \frac{K}{4} x_1 = f$

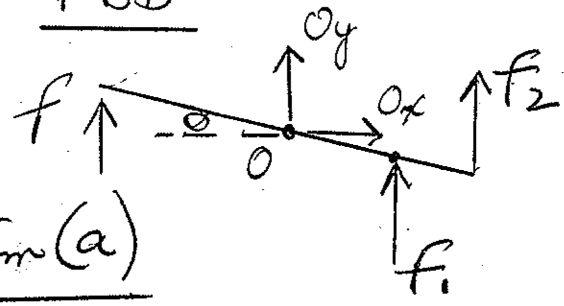
pulley 1 has mass: $\left[m_1 + \frac{I_p}{R^2}\right] \ddot{x}_1 + \frac{K}{4} x_1 = f$

The attached mass m_1 now has its mass increased (effectively) by $\frac{I_p}{R^2}$.

We could have seen this immediately since

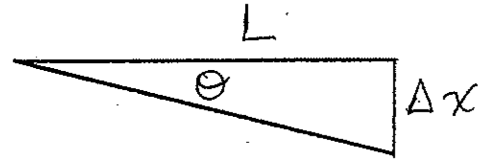
$R\theta = x_1 \qquad \theta = \frac{1}{R} x_1 \qquad m_1 \rightarrow m_1 + \frac{I_p}{R^2}$

7. FBD



System (a)

Fixed-Axis Rotation:



$\tan\theta = \dfrac{\Delta x}{L}$

$\Delta x = L \tan\theta$

$= L \dfrac{\sin\theta}{\cos\theta}$

$\cong L \dfrac{\theta}{1} = L\theta$

for small θ

Assumptions:

- frictionless pivot
- massless, rigid lever
- small displacements
- $\theta = 0$ springs are unstretched

$\Sigma M_0 = I_0 \ddot{\theta}$ But $I_0 = 0$

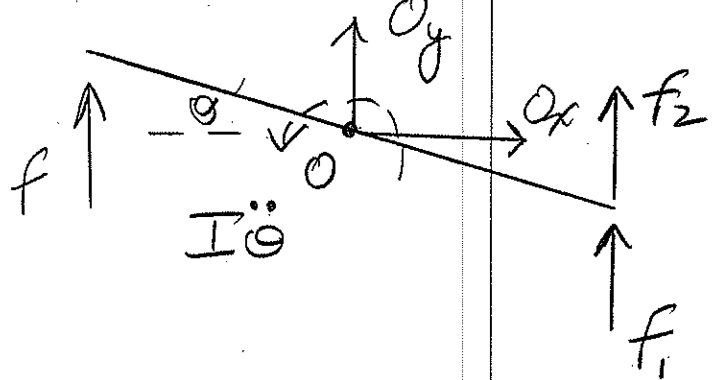
$\Sigma M_0 = 0$

$f L_1 - f_1 L_2 - f_2 L_3 = 0$

$f L_1 - (K_1 L_2 \theta) L_2 - (K_2 L_3 \theta) L_3 = 0$

$\underline{f L_1 = (K_1 L_2^2 + K_2 L_3^2)\theta}$

<u>System (b)</u> <u>FBD</u>



Now lever has mass with $I_o = I$

$$\Sigma M_o = 0$$

$$fL_1 - f_1 L_2 - f_2 L_2 - I\ddot{\theta} = 0$$

$$I\ddot{\theta} + f_1 L_2 + f_2 L_2 = fL_1$$

$$I\ddot{\theta} + (KL_2\theta)L_2 + (cL_2\dot{\theta})L_2 = fL_1$$

$$\underline{I\ddot{\theta} + cL_2^2\dot{\theta} + KL_2^2\theta = fL_1}$$

<u>Equation of Motion</u>