Linearization of Nonlinear Physical Effects

- Many real-world nonlinearities involve a "smooth" curvilinear relation between an independent variable *x* and a dependent variable *y*: $y = f(x)$
- A linear approximation to the curve, accurate in the neighborhood of a selected operating point, is the tangent line to the curve at this point.
- This approximation is given conveniently by the first two terms of the Taylor series expansion of *f(x)*:

$$y = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) + \left.\frac{d^2f}{dx^2}\right|_{x=\bar{x}} \frac{(x-\bar{x})^2}{2!} + \cdots$$

$$y \approx \bar{y} + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) \quad \Rightarrow \quad \begin{cases} y-\bar{y} \approx +\left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) \\ \hat{y} = K\hat{x} \end{cases}$$

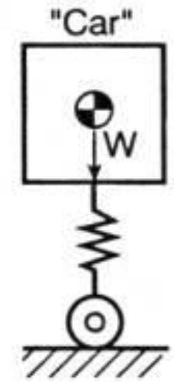
Linearization for a Nonlinear Spring

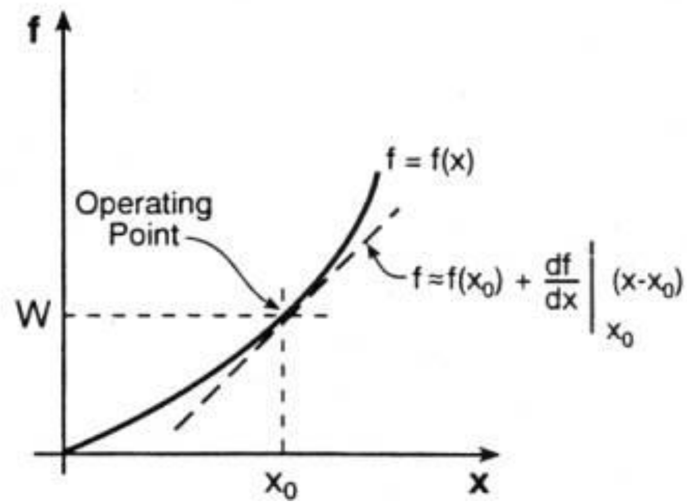
$$y = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) + \left.\frac{d^2f}{dx^2}\right|_{x=x_0}\frac{(x - x_0)^2}{2!} + \cdots$$

$$y \approx y_0 + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$

$$y - y_0 \approx +\left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$

$$\hat{y} = K\hat{x}$$





- For another example, in liquid-level control systems, when the tank is not prismatic, a nonlinear volume/height relationship exists and causes a nonlinear system differential equation. For a conical tank of height *H* and top radius *R* we would have:

$$V = \frac{\pi R^2}{3H^2} h^3$$

$$V \approx \frac{\pi R^2 \bar{h}^3}{3H^2} + \frac{\pi R^2 \bar{h}^2}{H^2} \hat{h}$$

- Often a dependent variable y is related nonlinearly to several independent variables x_1, x_2, x_3, etc. according to the relation: $y=f(x_1, x_2, x_3, \ldots)$.

- We may linearize this relation using the multivariable form of the Taylor series:

$$y \approx f\left(\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots\right) + \left(\left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \left(x_1 - \bar{x}_1\right) \right) + \left(\left.\frac{\partial f}{\partial x_2}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \left(x_2 - \bar{x}_2\right) \right)$$

$$+ \left(\left.\frac{\partial f}{\partial x_3}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \left(x_3 - \bar{x}_3\right) \right) + \cdots$$

$$y \approx \bar{y} + \left(\left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \hat{x}_1 \right) + \left(\left.\frac{\partial f}{\partial x_2}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \hat{x}_2 \right) + \left(\left.\frac{\partial f}{\partial x_3}\right|_{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots} \hat{x}_3 \right) + \cdots$$

$$\hat{y} = K_1 \hat{x}_1 + K_2 \hat{x}_2 + K_3 \hat{x}_3 + \cdots$$

The partial derivatives can be thought of as the sensitivity of the dependent variable to small changes in that independent variable.

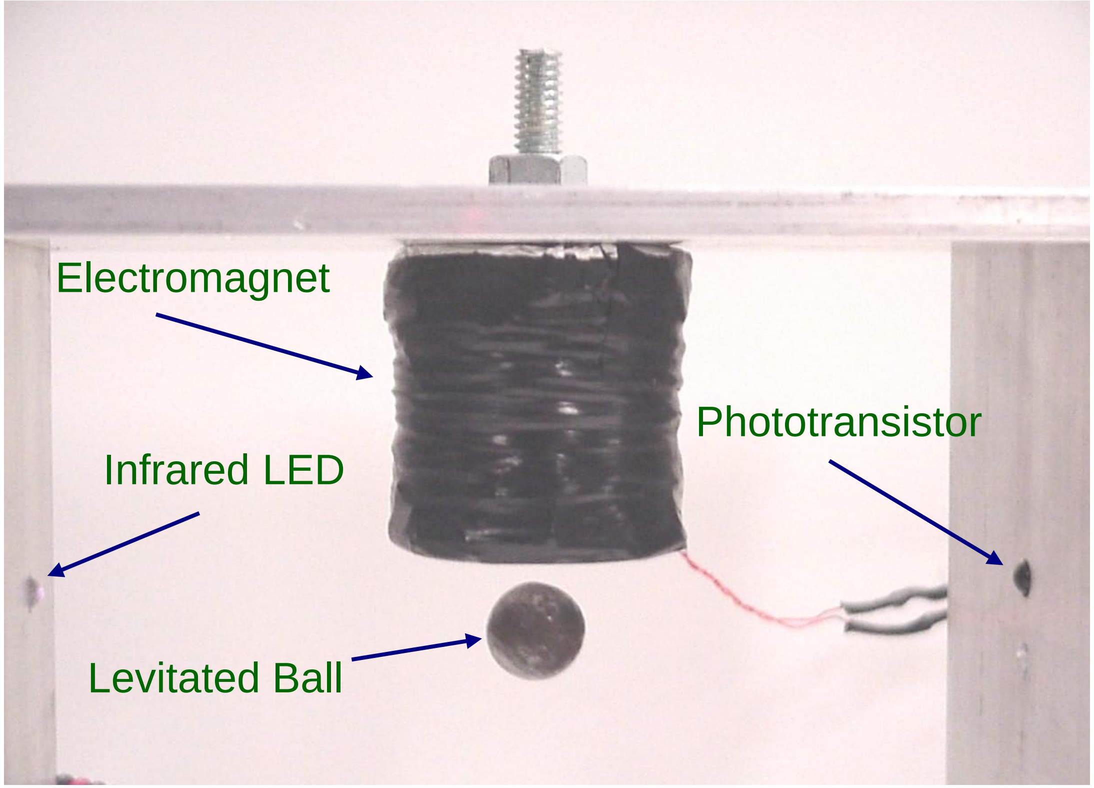
- For example, in a ported gas-filled piston/cylinder where gas mass, temperature, and volume are all changing, the perfect gas law gives us for pressure *p*:

$$p = \frac{RTM}{V}$$

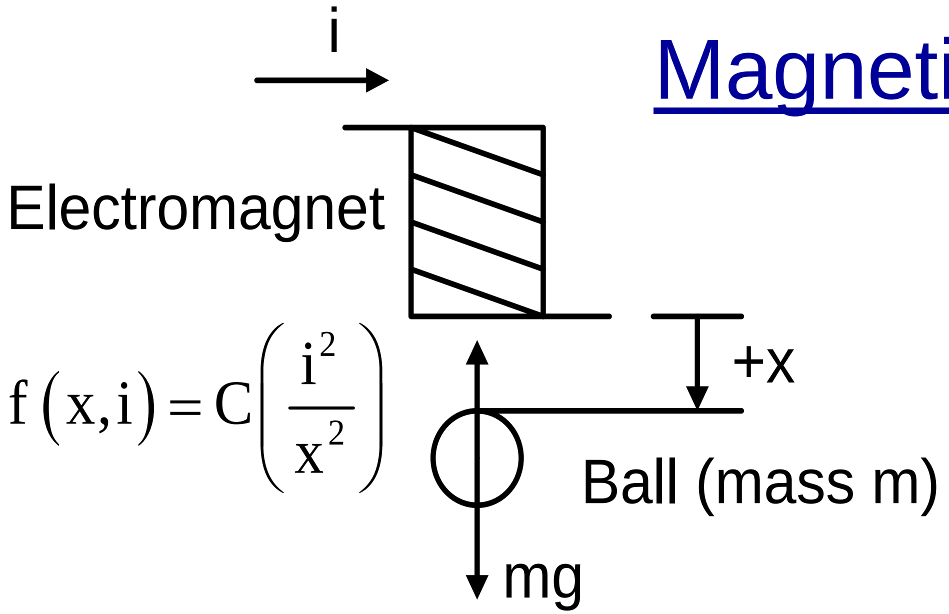
$$p \approx \frac{R\bar{T}\bar{M}}{\bar{V}} + \frac{R\bar{M}}{\bar{V}}\left(T - \bar{T}\right) + \frac{R\bar{T}}{\bar{V}}\left(M - \bar{M}\right) - \frac{R\bar{M}\bar{T}}{\bar{V}}\left(V - \bar{V}\right)$$

Example: Magnetic Levitation System

Applications include magnetic bearings for vacuum pumps, conveyor systems in clean rooms, high-speed levitated trains, and electromagnetic automotive valve actuators.



Magnetic Levitation System



$$f(x,i) = C\left(\frac{i^2}{x^2}\right)$$

Equation of Motion:

$$m\ddot{x} = mg - C\left(\frac{i^2}{x^2}\right)$$

At Equilibrium:

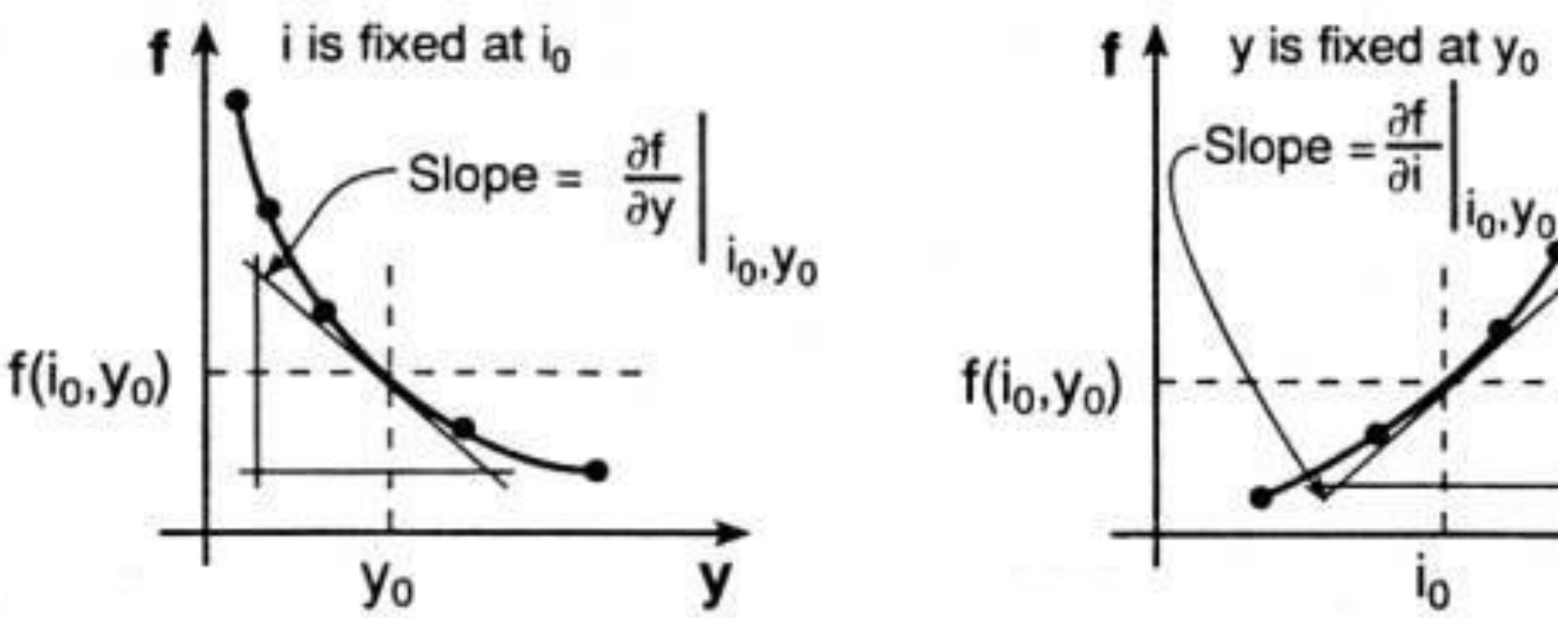
$$mg = C\left(\frac{\bar{i}^2}{\bar{x}^2}\right)$$

Linearization:

$$C\left(\frac{i^2}{x^2}\right) \approx C\left(\frac{\bar{i}^2}{\bar{x}^2}\right) - C\left(\frac{2\bar{i}^2}{\bar{x}^3}\right)\hat{x} + C\left(\frac{2\bar{i}}{\bar{x}^2}\right)\hat{i}$$

$$m\ddot{\hat{x}} = mg - C\left(\frac{\bar{i}^2}{\bar{x}^2}\right) + C\left(\frac{2\bar{i}^2}{\bar{x}^3}\right)\hat{x} - C\left(\frac{2\bar{i}}{\bar{x}^2}\right)\hat{i}$$

$$m\ddot{\hat{x}} = C\left(\frac{2\bar{i}^2}{\bar{x}^3}\right)\hat{x} - C\left(\frac{2\bar{i}}{\bar{x}^2}\right)\hat{i}$$



Use of Experimental Testing in Multivariable Linearization

$$f_m = f(i, y)$$

$$f_m \approx f(i_0, y_0) + \left.\frac{\partial f}{\partial y}\right|_{i_0, y_0} (y - y_0) + \left.\frac{\partial f}{\partial i}\right|_{i_0, y_0} (i - i_0)$$