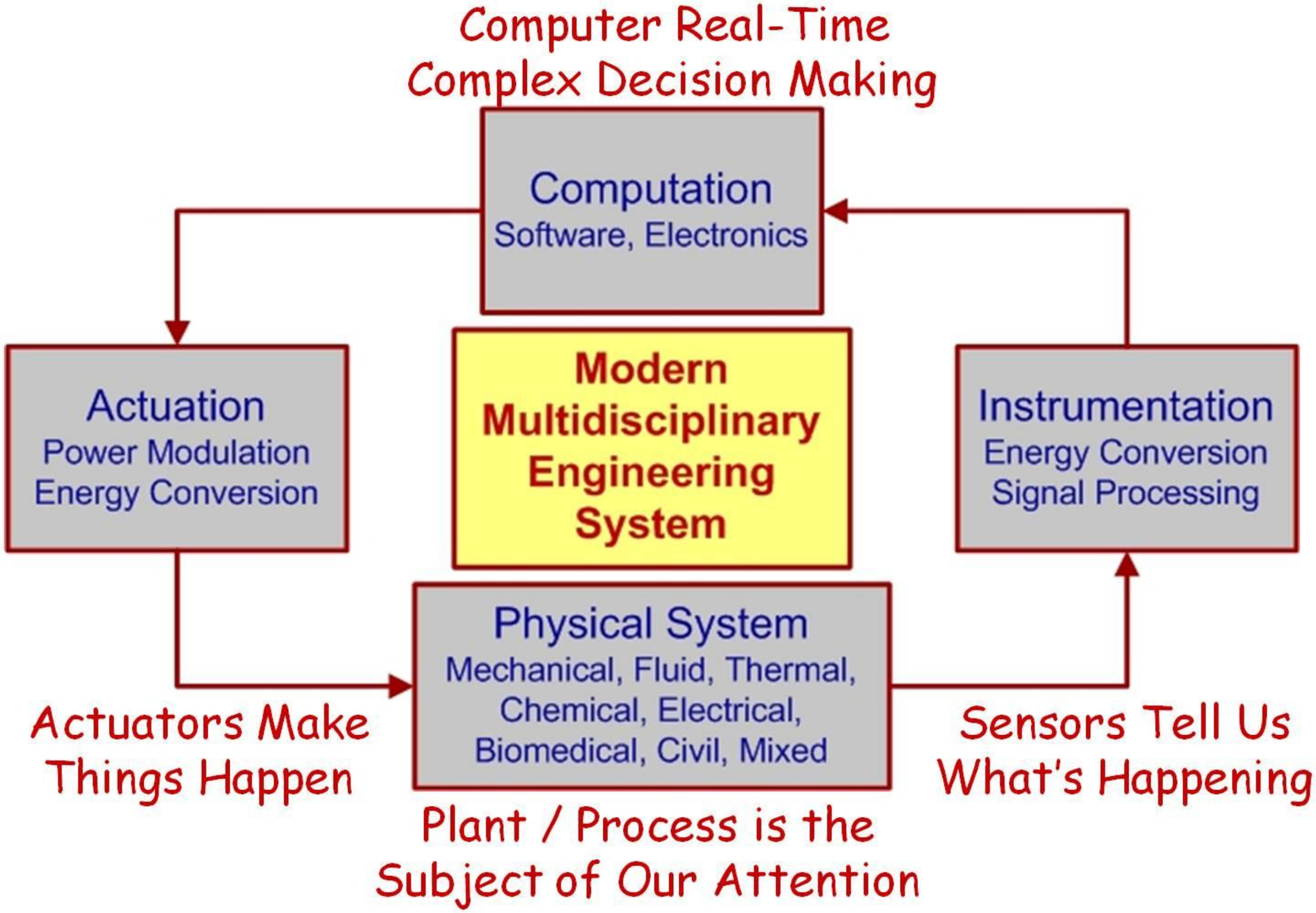
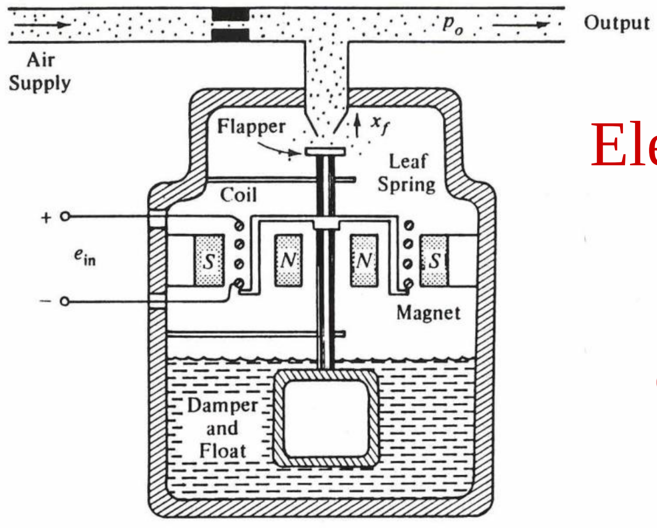
Electrical Systems:
Modeling, Analysis, Measurement, & Control



Introduction and Overview

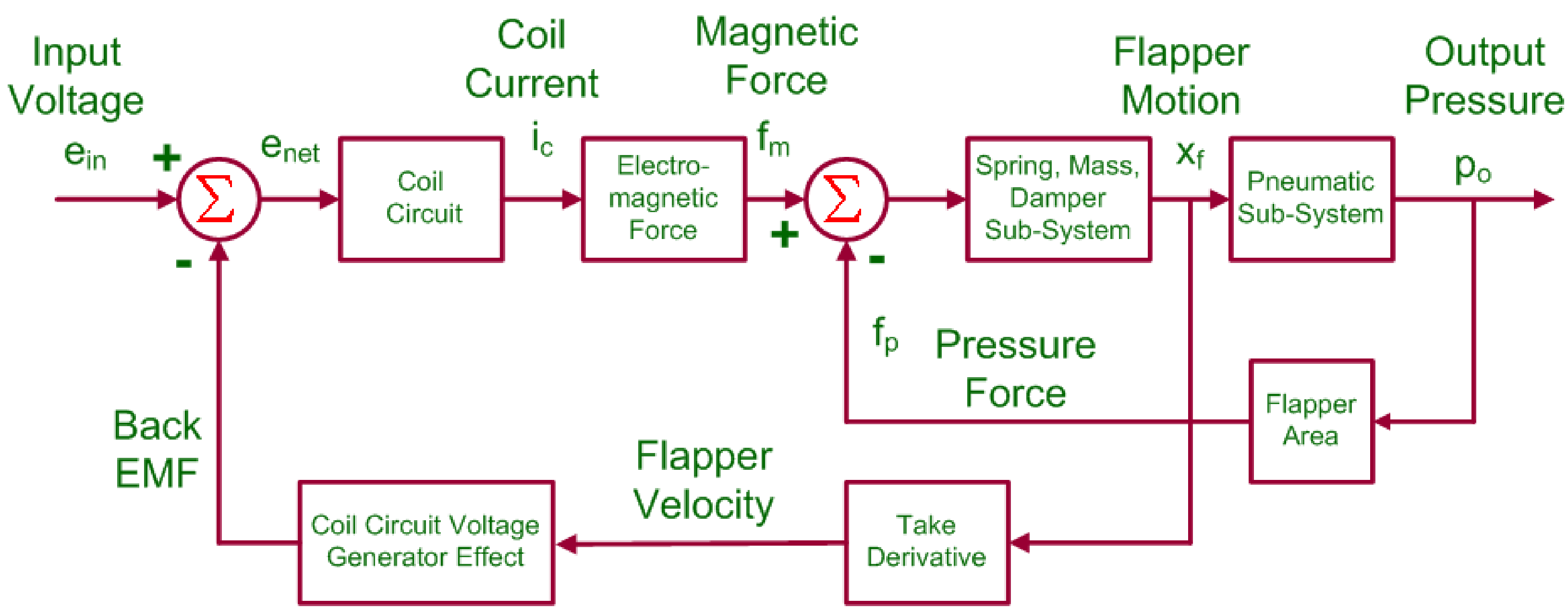
- Engineering Systems
- Systems Inputs for Analysis and Measurement
  - Step Input → Step Response
  - Sine Input → Frequency Response
- Physical Modeling
- Time Domain and Frequency Domain
- **Note**: *These topics are discussed in detail in the supplementary notes provided. These notes are **essential reading**.*
  - *Engineering Systems*
  - *System Inputs, Physical Modeling, Time & Frequency Domains*

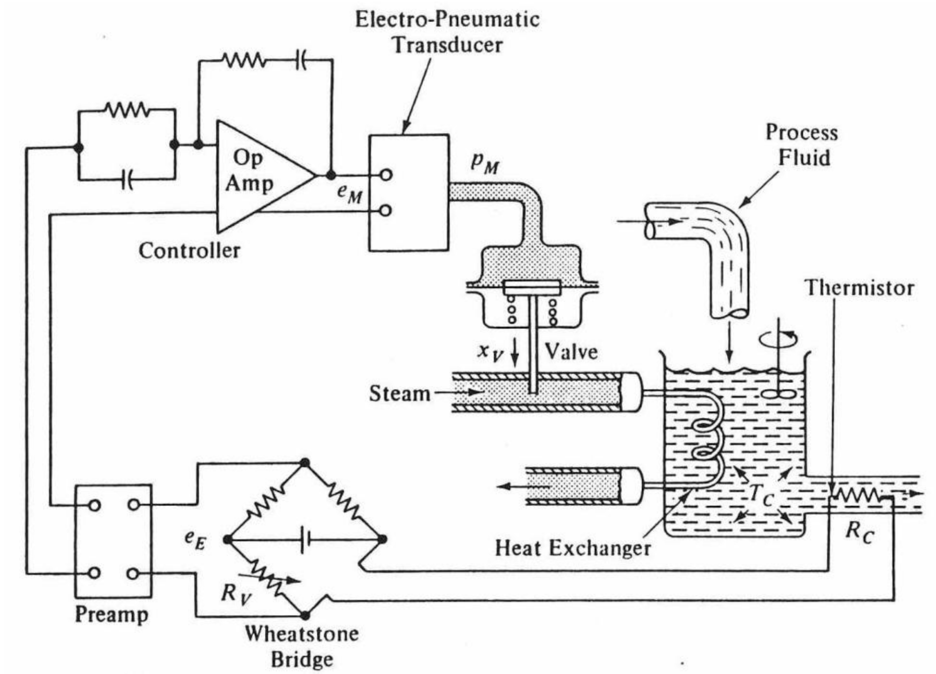


Electro-Pneumatic Transducer: An Engineering System

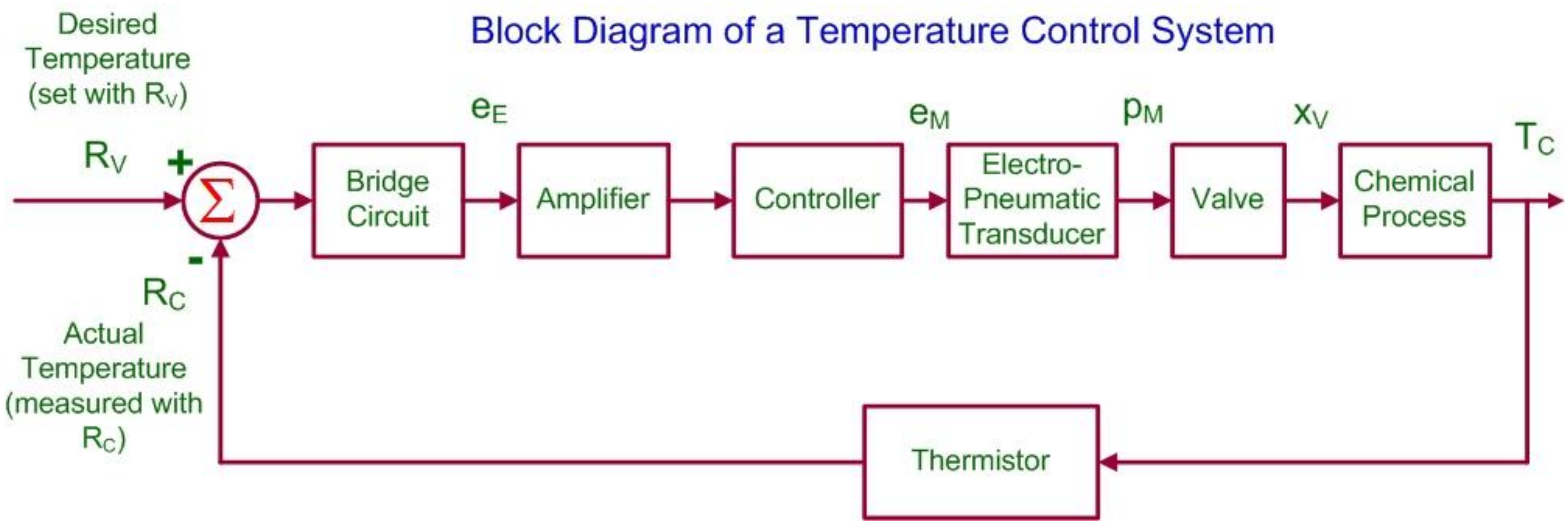
Note the three methods of engineering communication: picture, schematic, & block diagram!

Block Diagram of an Electro-Pneumatic Transducer



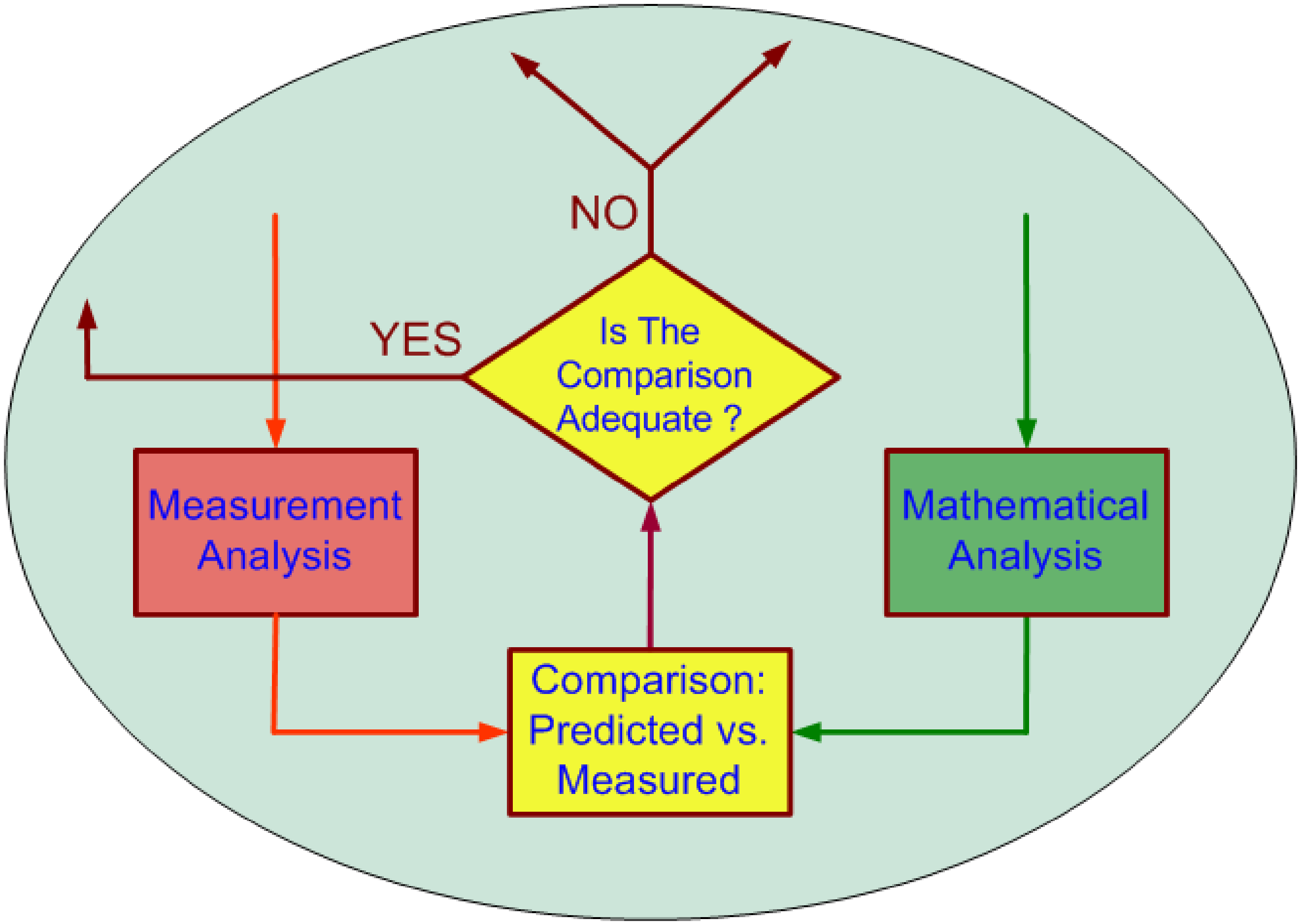


Temperature Feedback Control System: A Larger-Scale Engineering System

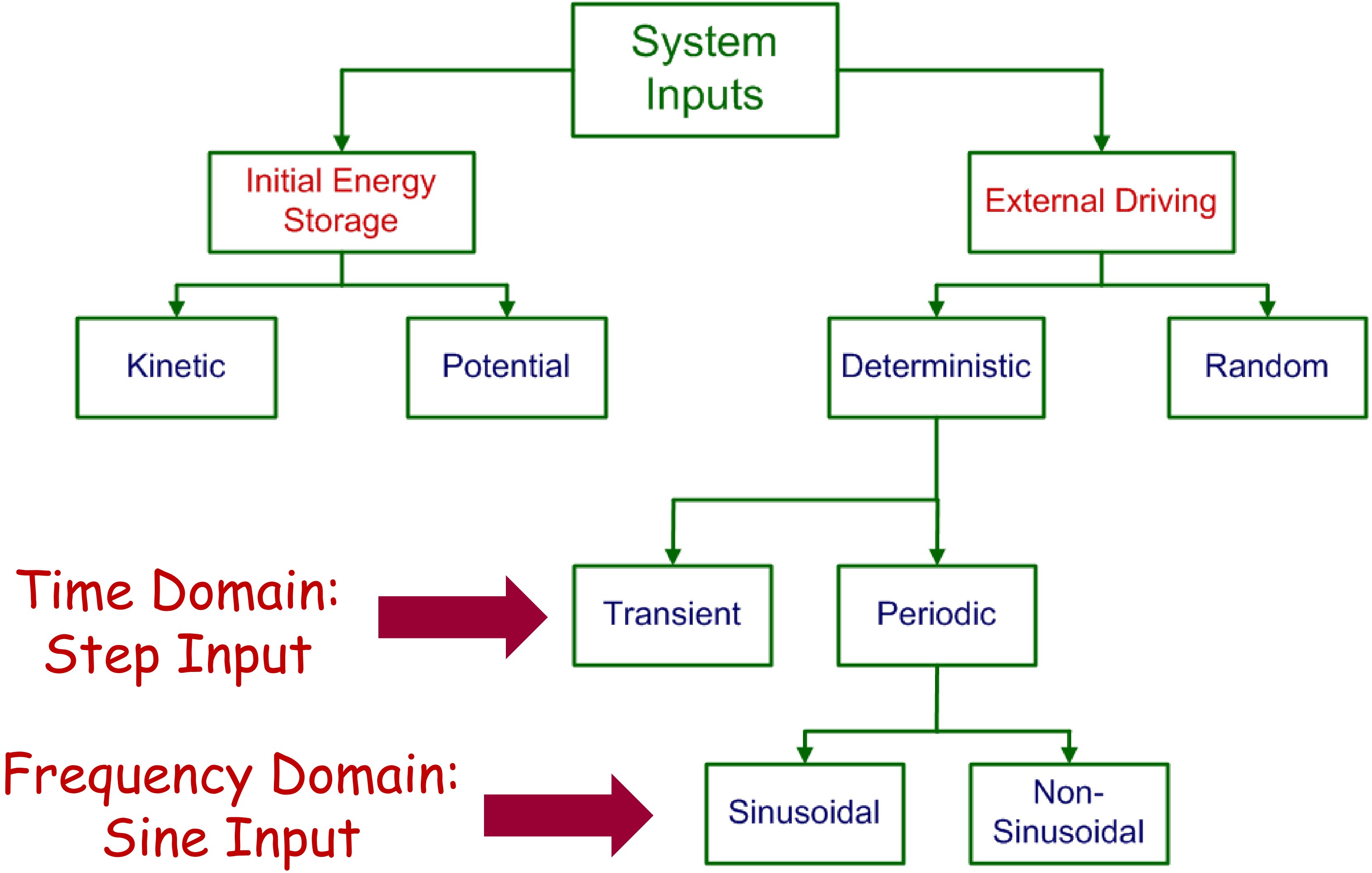


Questions:

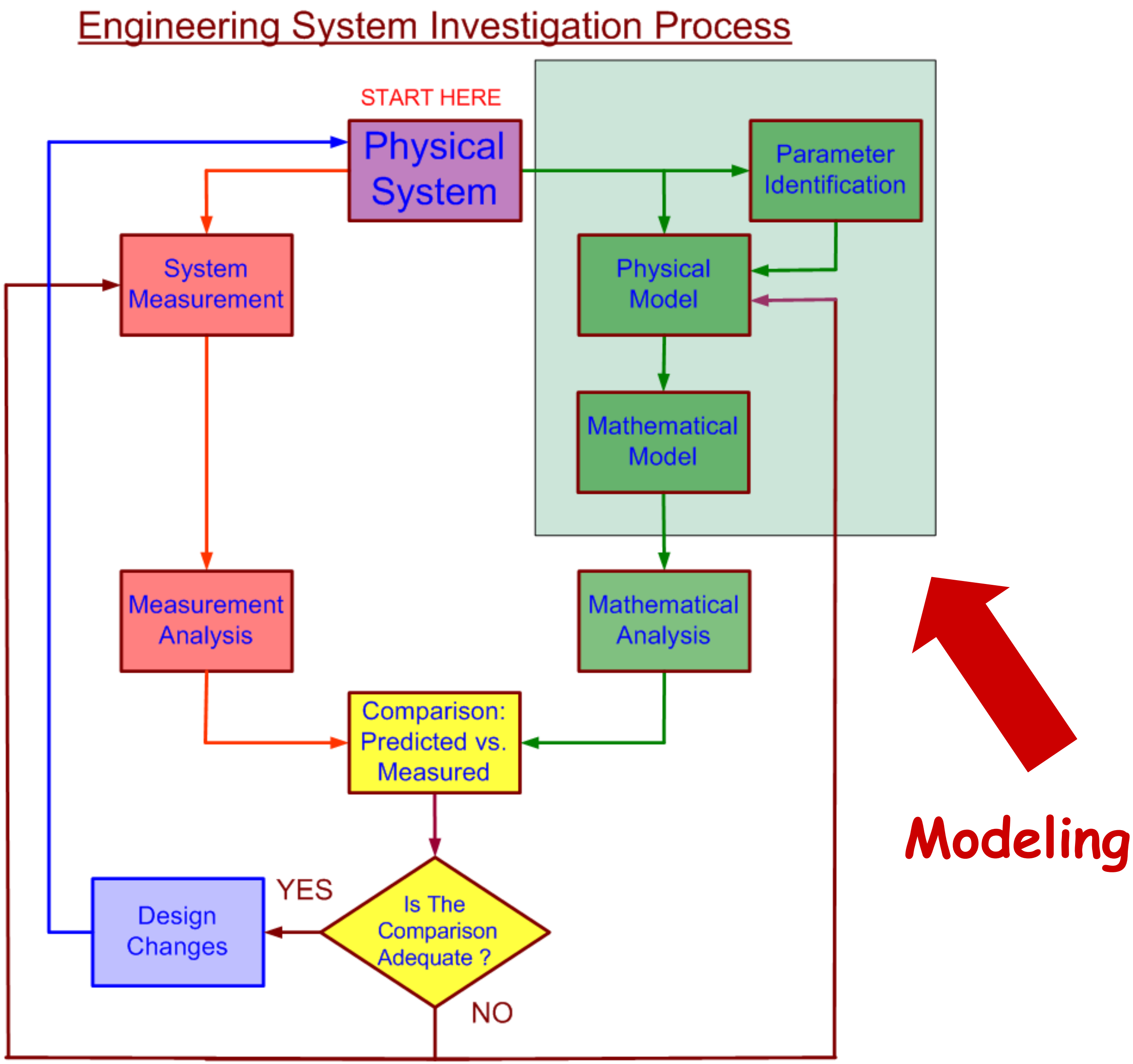
What is the input to the actual physical system that we are measuring the response to?
What is the input to the mathematical model that we are predicting the response to?
Of course, these two inputs must be the same if we are to compare the measured response to the predicted response.
Are there standard inputs used by engineers in the investigation process?
If so, what are they? Why are they effective? Why not use the actual real-world inputs?



Classification of System Inputs



Engineering System Investigation Process

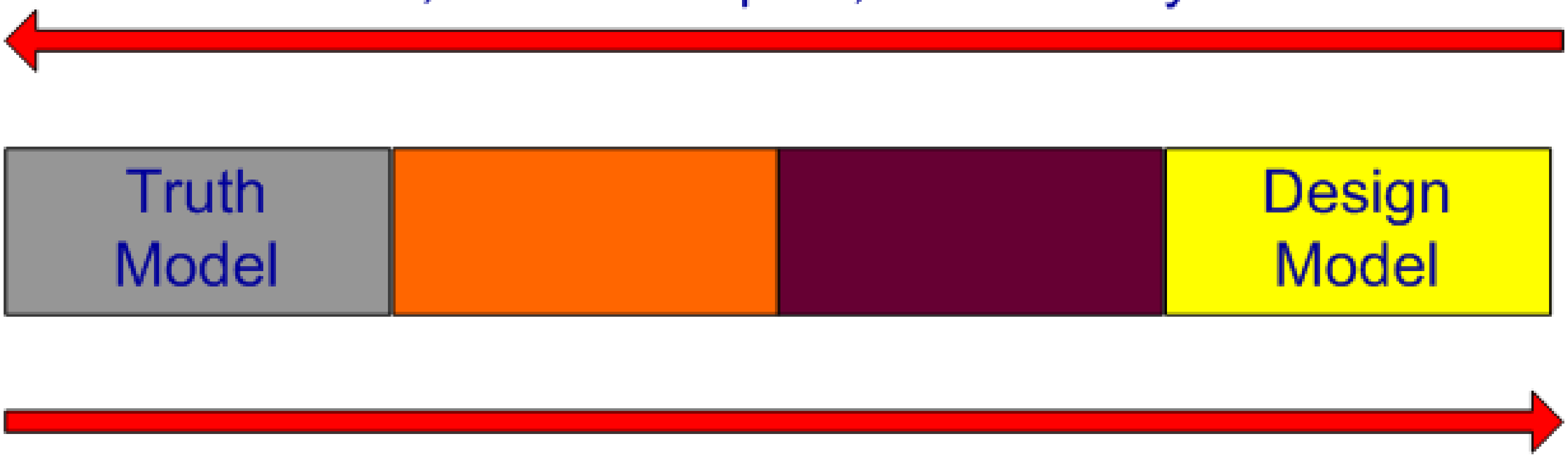


Physical Modeling

Hierarchy of Models

Always Ask: Why Am I Modeling ?

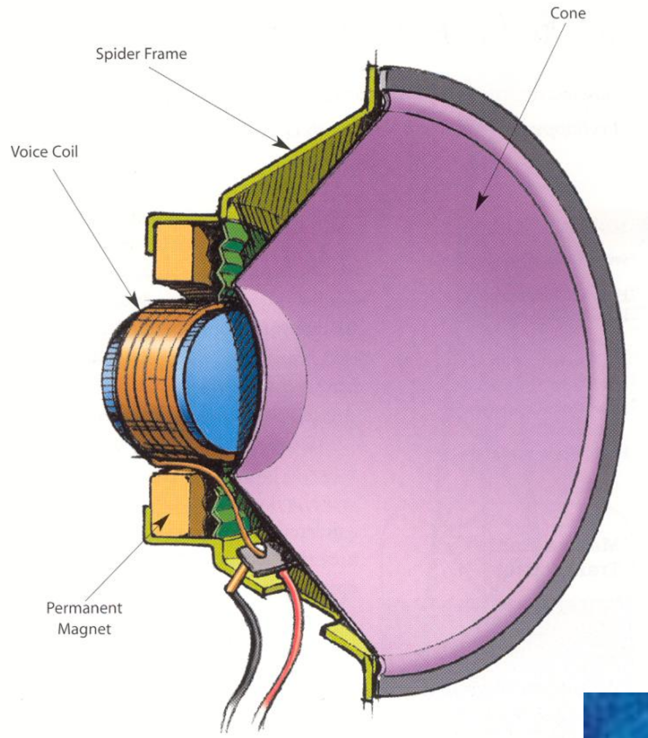
More Real, More Complex, Less Easily Solved



Less Real, Less Complex, More Easily Solved

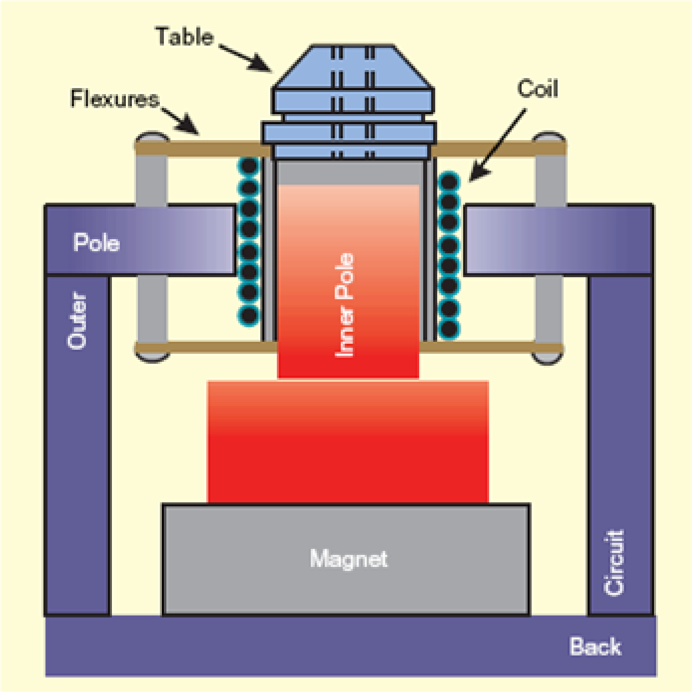
Purpose of Modeling is INSIGHT !

Loudspeaker

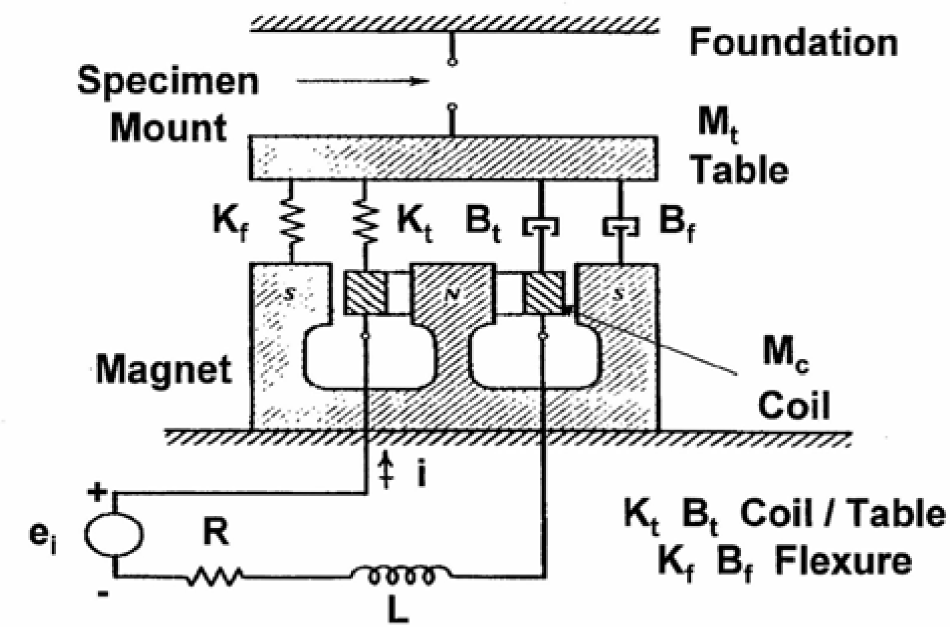


Electro-Dynamic Vibration Exciter

Electro-Dynamic Vibration Exciter
Physical System vs. Physical Model



Physical System

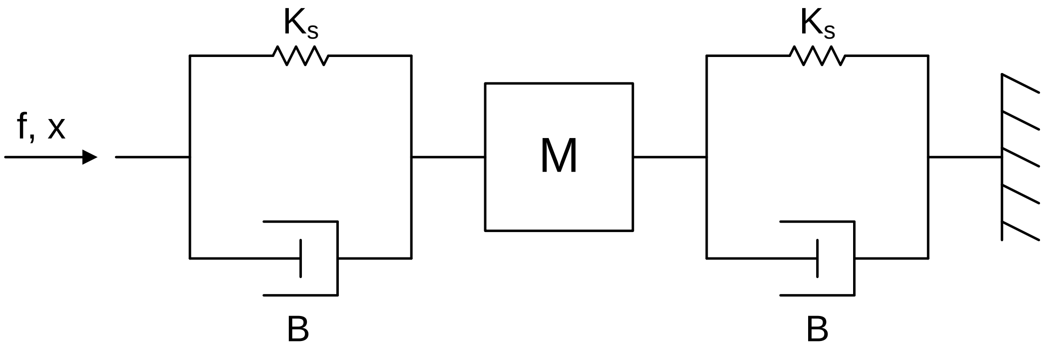


Physical Model

Pure and Ideal Elements

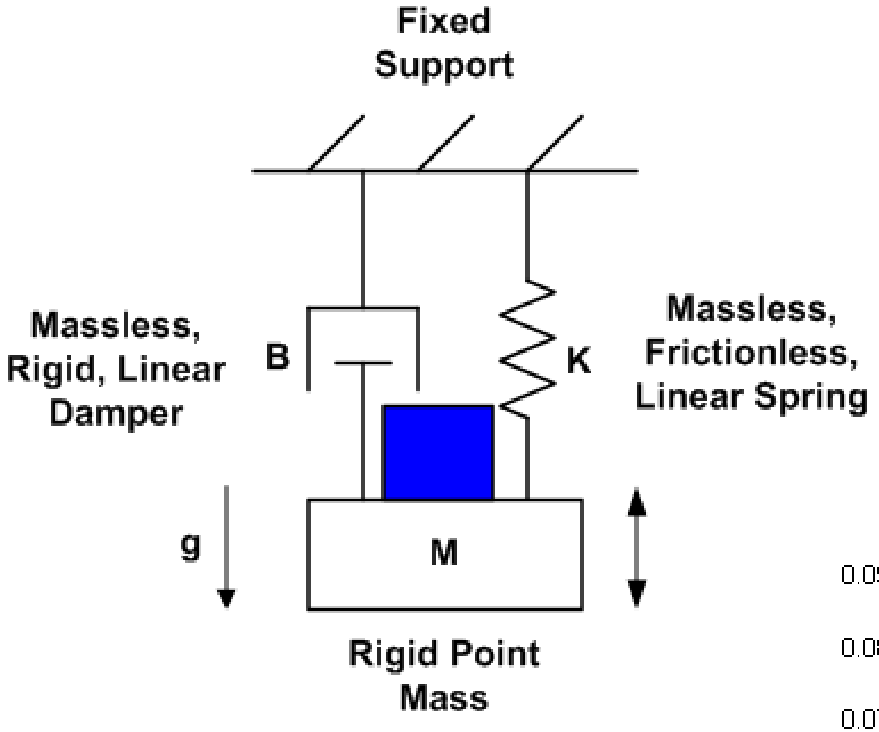
Real Spring
Physical Device vs. Physical Model

Physical System



Physical Model

Pure and Ideal Elements



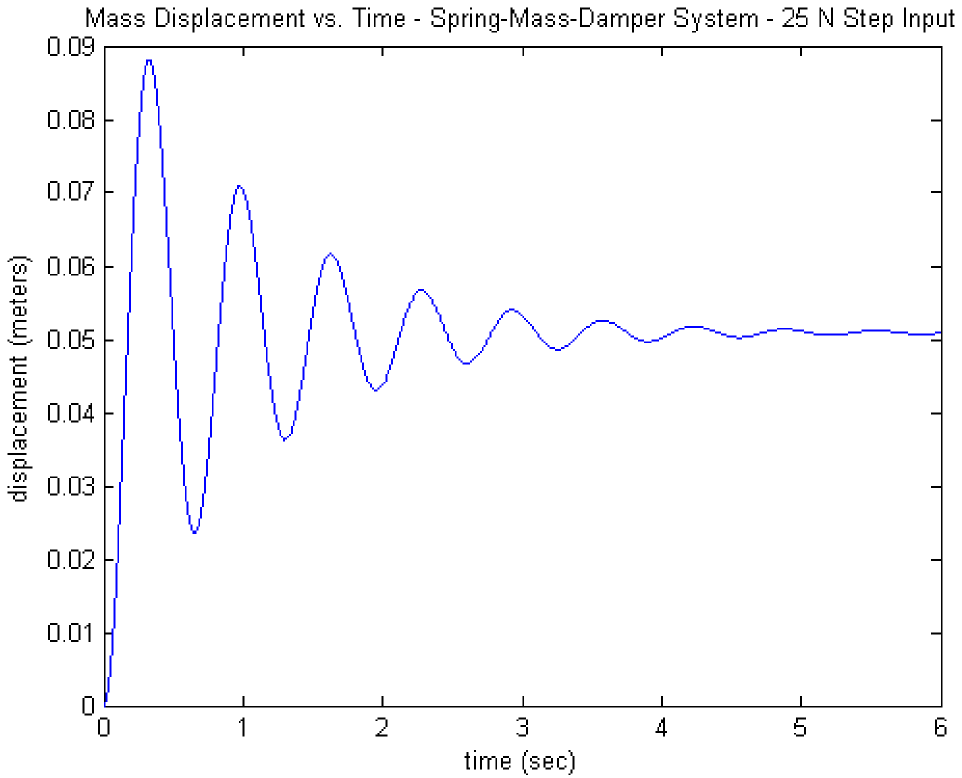
Time Domain

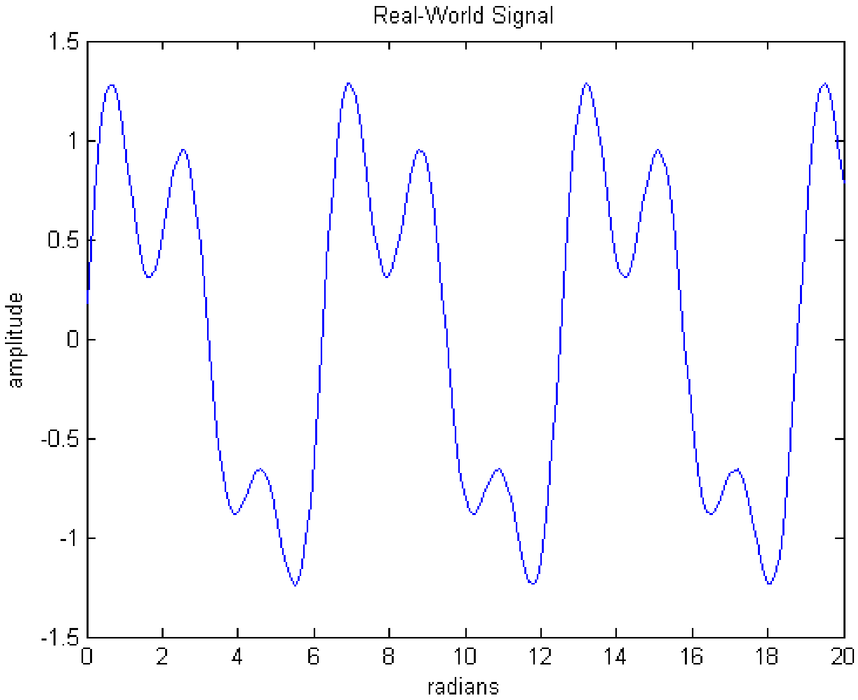
We measure how long something takes.

Step Input

The system is "at rest" at time t = 0 and we instantly change the input quantity, from wherever it was just before t = 0, by a given amount, either positive or negative, and then keep the input constant at this new value "forever."

Step Response

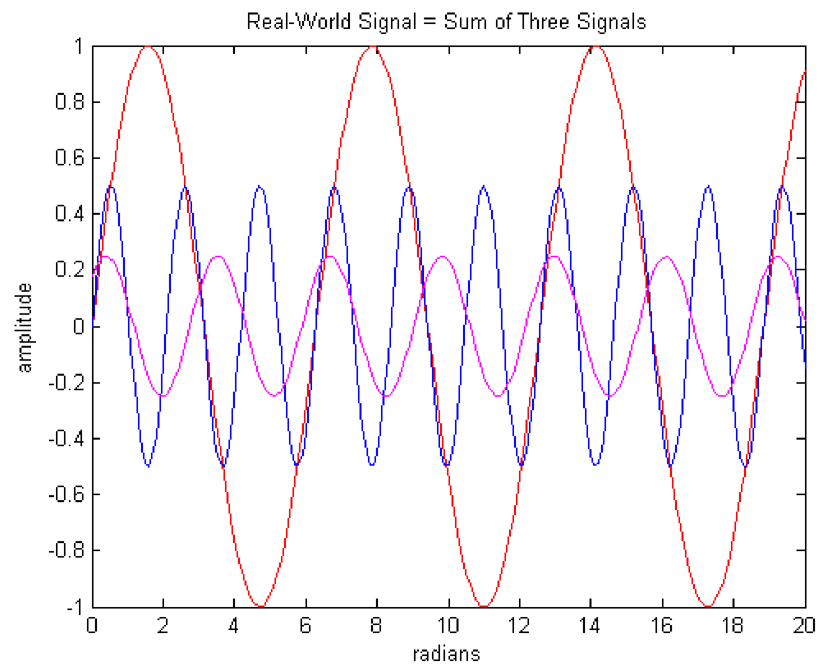




Frequency Domain

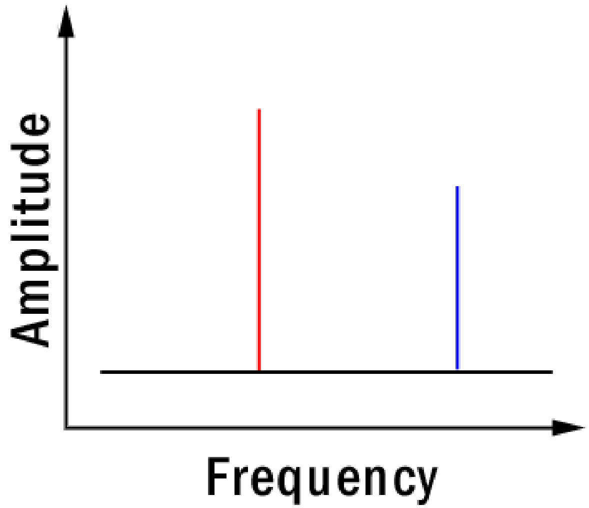
We measure how fast or slow things are happening.

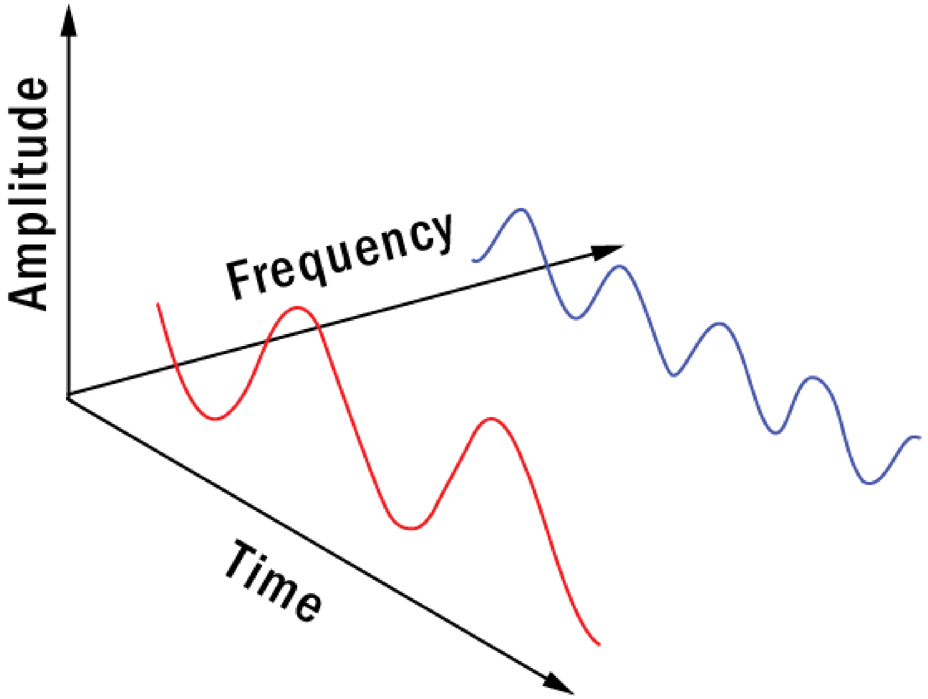
Frequency Response

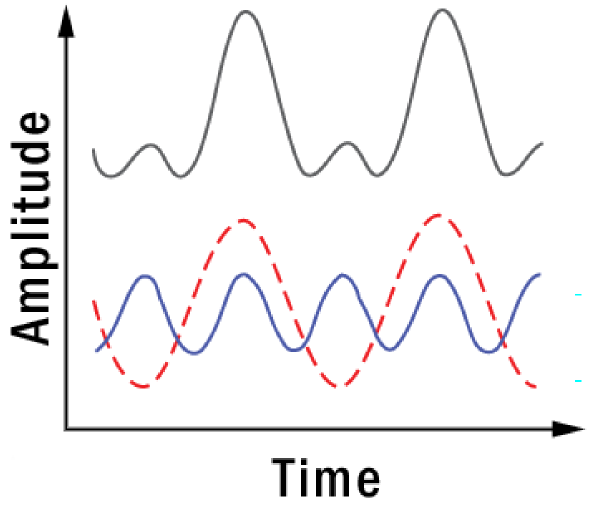


Any real-world signal can be broken down into a sum of sine waves of varying frequency, amplitude, and phase, and this combination of sine waves is unique.

Time & Frequency Domains







Why Study Electrical Systems First?

- Why do we start with electrical systems?
  - All engineering systems built today have electronics and controls (either analog or digital with a microcontroller) integrated into the design from the very beginning of the design process – after-thought add-ons are not permitted as they lead to less-than-optimum designs.
  - Many modeling and analysis concepts and techniques are easier to explain and easier to grasp using electrical elements.
  - Numerous tools and methods for analyzing electrical circuits have been developed over the years and all these can be applied to the modeling and analyzing of mechanical, magnetic, fluid, thermal, and multidisciplinary systems as well.

Electrical System Topics

- Part 1
  - Voltage, Current, Power
  - Resistor, Impedance, and Potentiometer
  - Kirchhoff's Circuit Laws: KVL and KCL
  - Current and Voltage Sources & Meters: Ideal and Real
  - Circuit Loading
  - Op-Amp as a Buffer
  - Norton and Thevenin Equivalent Circuits
  - Impedance Matching

Voltage, Current, Power

- Voltage and Current
  - The concepts of voltage and current are used in electrical and electronics engineering to describe the behavior of engineering systems that use electricity.
- Voltage
  - Voltage, V or e (volts), also called potential difference or electromotive force (emf), is the amount of work done or the energy (joules) required in moving a unit of positive charge (one coulomb) from a negative point (lower potential) to a more positive point (higher potential).

– 1 volt (V) = 1 joule (J) per coulomb (C). The unit for voltage is volts and the unit for electrical charge is coulomb. One coulomb of electrical charge is equal in magnitude to 6.22 x 10^{18} electrons. $V = J / C$

– A voltage is the measure of the difference in potential across a component; you need two points to measure a voltage.

– Often we use a common standard reference point with a potential of zero volts called an earth point or ground.

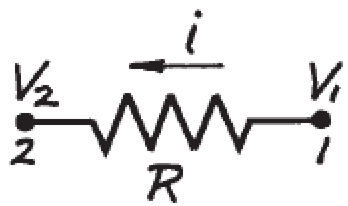
– Shown is a definition sketch of an electrical element, a resistor:

– If V_2 is not equal to V_1, then electrical charge flows from one side of the element to the other. This flow of electrical charge per unit time is called current (symbol i) and is measured in amperes or amps.

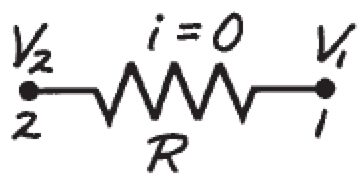
- Current
  - Current, i (amperes), is the amount of electric charge (coulombs) flowing past a specific point in a conductor over an interval of one second.
  - 1 ampere (A) = 1 coulomb (C) per second (s) $A = C/s$
  - Electric current is created by the flow of negatively charged electrons. By convention, we choose positive current to be in the opposite direction to the flow of electrons.
  - Current is a flow of electrons into, out of, or through a component.

- We use the symbol V_{21} to represent $V_2 - V_1$.
- Shown is a definition sketch of an electrical element with different voltage / current values.

$V_2 > V_1$



$V_2 < V_1$



$V_2 = V_1$

- Concepts of Work, Power, and Energy in Electrical Elements
  - Voltage is defined as the work that must be done to move a unit of electrical charge from one point to another, from point 1 to point 2 in this case:

$$V_{21} = \frac{dW_{21}}{dq}$$

  - The unit of measure for work is the joule. Volts (V) are then joules (J) per coulomb (C), i.e., one volt is equal to one joule of work per coulomb of charge. One joule of work is equivalent to 0.737 ft-lbs.
  - Current is the flow of electrical charge per unit time.

$$i = \frac{dq}{dt}$$

– Power, P, is defined as the rate at which work is performed.

$$P = \frac{dW}{dt}$$

– Power, P, measured in watts (W), is the rate at which energy is used by an element.

– The product of voltage differential across an electrical element and the current flowing through the element is equal to power.

– Power = Voltage x Current = V x i

$$V_{21} \times i = \frac{dW_{21}}{dq} \times \frac{dq}{dt} = \frac{dW_{21}}{dt} = P$$

$$W = \frac{J}{C}\frac{C}{s} = \frac{J}{s}$$

$$W = VA$$

– Since work is a form of energy E, we can write:

$$\frac{dE}{dt} = P$$

– This equation can be integrated to obtain the energy stored in, or dissipated by, an electrical element over a time interval from $t = t_1$ to $t = t_2$:

$$E = \int P dt$$

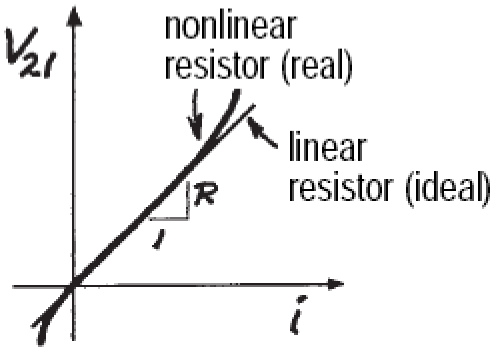
$$E_2 - E_1 = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} (V_{21} i)\,dt$$

The Resistance Element

- This is the most common of all electrical elements.
- It is intentionally or unintentionally present in every real electrical system.
- Pure and ideal resistance element has a mathematical model:

$$i = \frac{e}{R}$$

- Value of R is given in ohms (Ω).
- Strict linearity between *e* and *i*
- Instantaneous response of *i* to *e* or *e* to *i*
- All electrical energy supplied is dissipated into heat.
- Real Resistors
  - Non-ideal (not exactly linear)



- Impure – they exhibit some capacitance and inductance effects which make themselves known only when current and voltage are changing with time.
- A steady-state experiment will reveal departures from ideal behavior, but will not reveal impurity of a resistor.
- When measuring the resistance of a resistor with an ohmmeter, the R value obtained is good for the one value of current that the ohmmeter is using to measure the resistance.

– A crude check of linearity can quickly be made by measuring the resistance of a resistor on several ranges of an ohmmeter. If the indicated value is significantly different when measured on different ranges, the resistor is nonlinear.

– Definition of Resistance R (ohms) and Conductance G (siemens)

$$R = \frac{e}{i} \qquad G = \frac{i}{e}$$

– The energy delivered to a resistor during a time interval is given by:

$$E_2 - E_1 = \int_{t_1}^{t_2} (V_{21} i)\,dt = \int_{t_1}^{t_2} V_{21}\left(\frac{V_{21}}{R}\right) dt = \frac{1}{R}\int_{t_1}^{t_2} V_{21}^2\,dt$$

$$= \int_{t_1}^{t_2} (V_{21} i)\,dt = \int_{t_1}^{t_2} (Ri)i\,dt = R\int_{t_1}^{t_2} i^2 dt$$

– A resistor always dissipates power, regardless of the direction of the current or the sign of the voltage. Energy cannot be retrieved from a resistor. This element only dissipates energy.

– Instantaneous electric power P is given by:

$$P = ei = i(iR) = i^2R = e\frac{e}{R} = \frac{e^2}{R} = e^2G$$

– Power is always positive; the resistor always takes power from the source supplying it.

– Since the resistor cannot return power to the source, all the power supplied is dissipated into heat.

– Electric power (watts) is the *heating rate* for the resistor.

- Internal heat generation causes the resistor temperature to rise.
  - When the resistor temperature is higher than that of its surroundings, heat transfer by conduction, convection, and radiation causes heat to flow away from the resistor.
  - When the resistor gets hot enough, this heat transfer rate just balances the e^2/R heat generation rate and the resistor achieves an equilibrium temperature somewhere above room temperature.
  - In a real resistor this temperature cannot be allowed to get too high, or else the R value changes excessively or the resistor may actually burn out.

- Instantaneous dynamic response is characteristic of a pure resistance element (zero-order dynamic system model).
- Real resistors are always impure and this prevents the instantaneous step response, and a frequency response with the perfectly flat amplitude ratio and the zero phase angle.
- Since practical systems always deal with a limited range of operation, if a real resistor behaves nearly like a pure/ideal model over its necessary range, the fact that it deviates elsewhere is of little consequence.
- Note that resistance elements can be pure without being ideal. For example a very useful nonlinear resistor is the semiconductor diode.

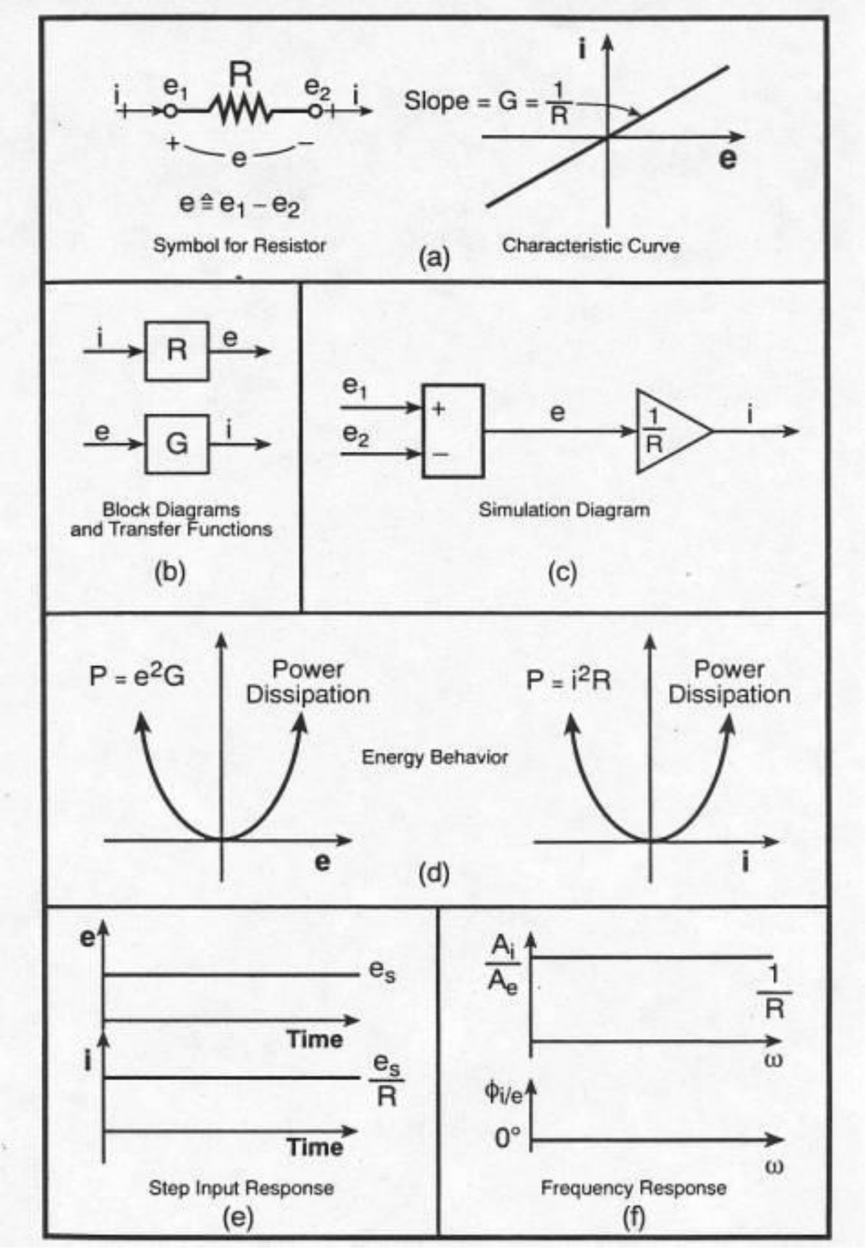
Pure & Ideal Resistance Element

$$i = \frac{e}{R} = Ge$$

$$e = Ri$$

Impedance Z

$$Z = \frac{\text{effort}}{\text{flow}} = \frac{\text{voltage}}{\text{current}} = \frac{e}{i} = R$$



Step Response

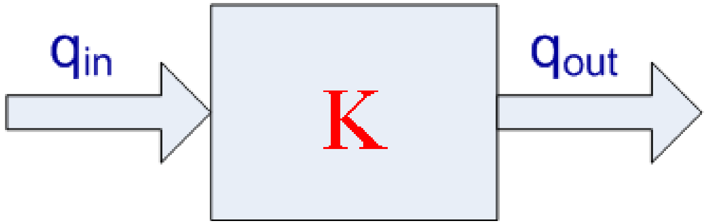
Resistor

$$i = \frac{1}{R} e$$

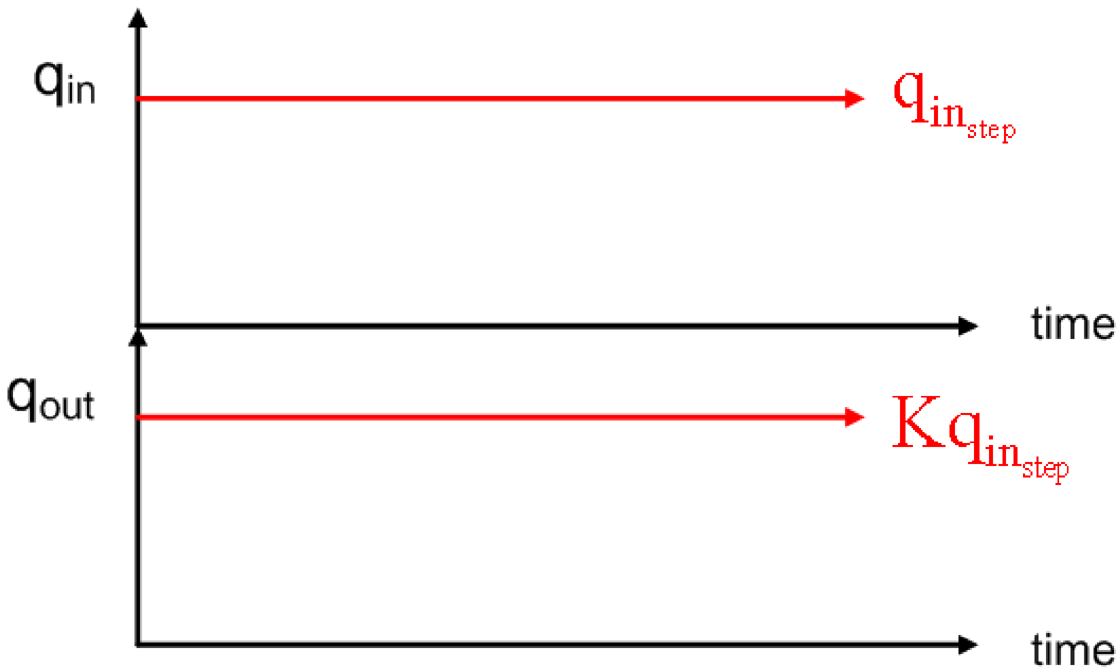
$$e = Ri$$

$q_{in} = i$

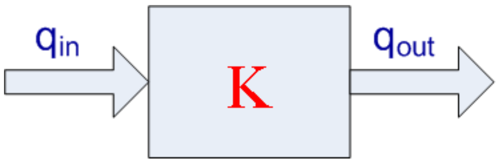
$q_{out} = e$



$$q_{out} = Kq_{in}$$



Frequency Response — Resistor



$$q_{out} = K q_{in}$$

$$q_{in} = A\sin(\omega t)$$

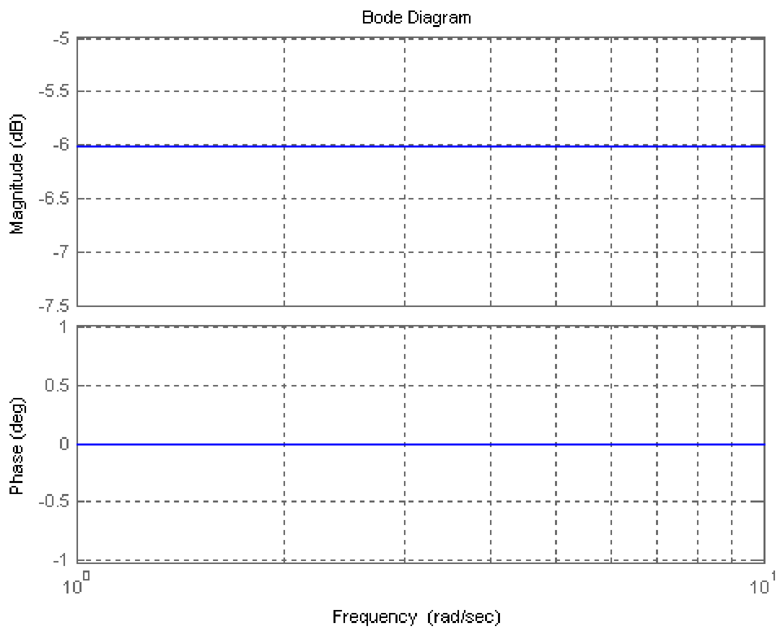
$$q_{out} = KA\sin(\omega t)$$

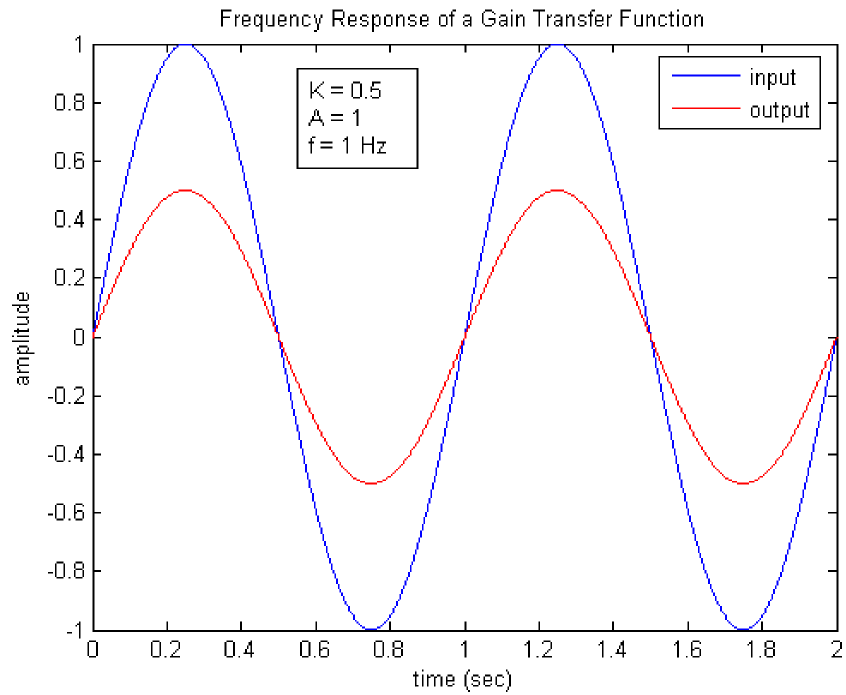
$$i = \frac{1}{R}e$$

$$e = Ri$$

$$q_{in} = i$$

$$q_{out} = e$$





- Resistors in series and parallel
  - If the same current passes through two or more resistors, those resistors are said to be in *series*, and they are equivalent to a single resistor whose resistance is the sum of the individual resistances. $R_{eq} = R_1 + R_2 + \cdots + R_i$
  - If the same voltage difference exists across two or more resistors, those resistors are said to be in *parallel* and they are equivalent to a single resistance whose reciprocal is equal to the sum of the reciprocals of the individual resistances.

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_i} \qquad R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \text{ Only for 2 resistors}$$

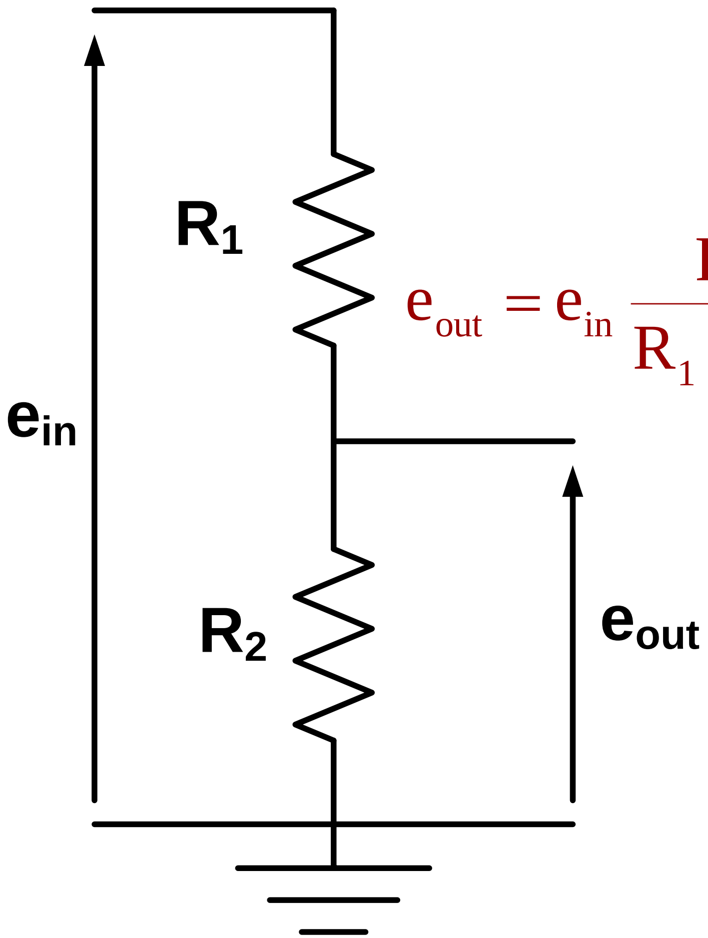
    - For resistances in parallel, the total parallel resistance is always dominated by, and is less than, the smallest resistance value in the circuit.

- Resistance of a Conductor
  - The resistance R of a conducting wire of length L, cross-sectional area A, and material resistivity ρ is given by:

$$R = \frac{\rho L}{A}$$

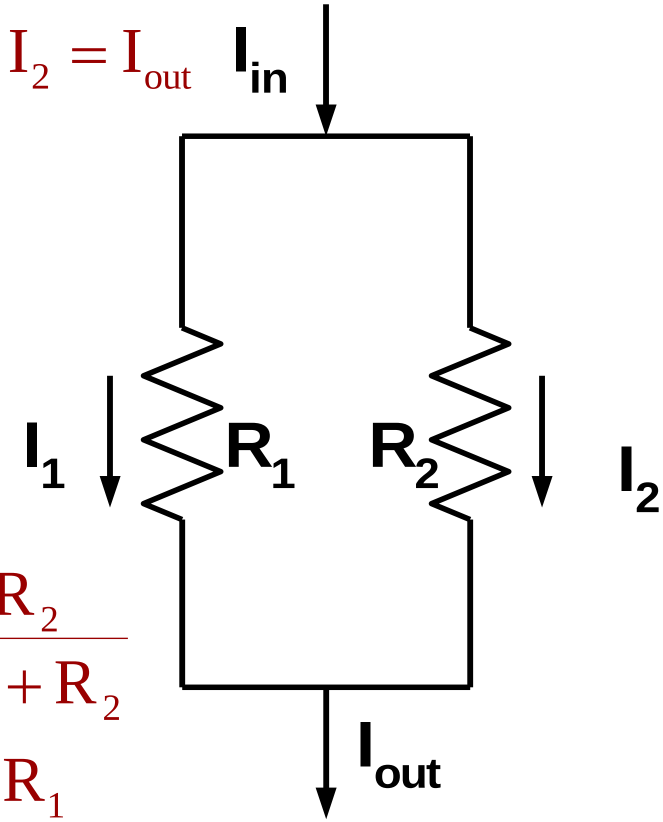
- Uses of Resistance in Circuits
  - Resistances are used in circuits to control or limit the amount of current flow in a circuit.
  - Resistances are used to convert a flow of current into a voltage. The resistor is the cheapest and simplest form of current-to-voltage converter available.
  - Resistances are used to reduce the size of a voltage from one circuit to another.

Voltage Divider and Current Divider



$$e_{out} = e_{in}\frac{R_2}{R_1 + R_2}$$

$$I_{in} = I_1 + I_2 = I_{out}$$



$$I_1 = I_{in}\frac{R_2}{R_1 + R_2}$$

$$I_2 = I_{in}\frac{R_1}{R_1 + R_2}$$

<u>Note</u>: R_1 and R_2 are in series

$$R_{eq} = R_1 + R_2$$

<u>Note</u>:
R1 and R2 are in parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

- Resistor Specifications
  - Resistance Value
  - Tolerance
    - Resistors are manufactured with well-defined tolerances, typically ranging from 10% to 0.1%; e.g., a 5% 47Ω resistor will have a value that departs by no more than ± 5% from 47Ω.
    - It is the mark of a good design not to use precision components except in key locations.
    - The normal, wide variations in resistor values set a limit on the accuracy needed for most calculations.
    - To achieve precision, the normal process is to combine a fixed resistor in series with a variable resistor and adjust the variable resistor to obtain the desired operating point.

– Power Rating
  - All resistors have power dissipation ratings, which tell how much power they can dissipate before their values change by more than their rated tolerance or before they fail.
  - Power-handling capability of commercially available resistors range from ¼ W to hundreds of watts. Low-power resistors are generally satisfactory for op-amp, transistor, and logic circuits; high-power resistors are needed in power supplies.
  - Check the power dissipation in designing a circuit and use resistors well below their rated power limits.

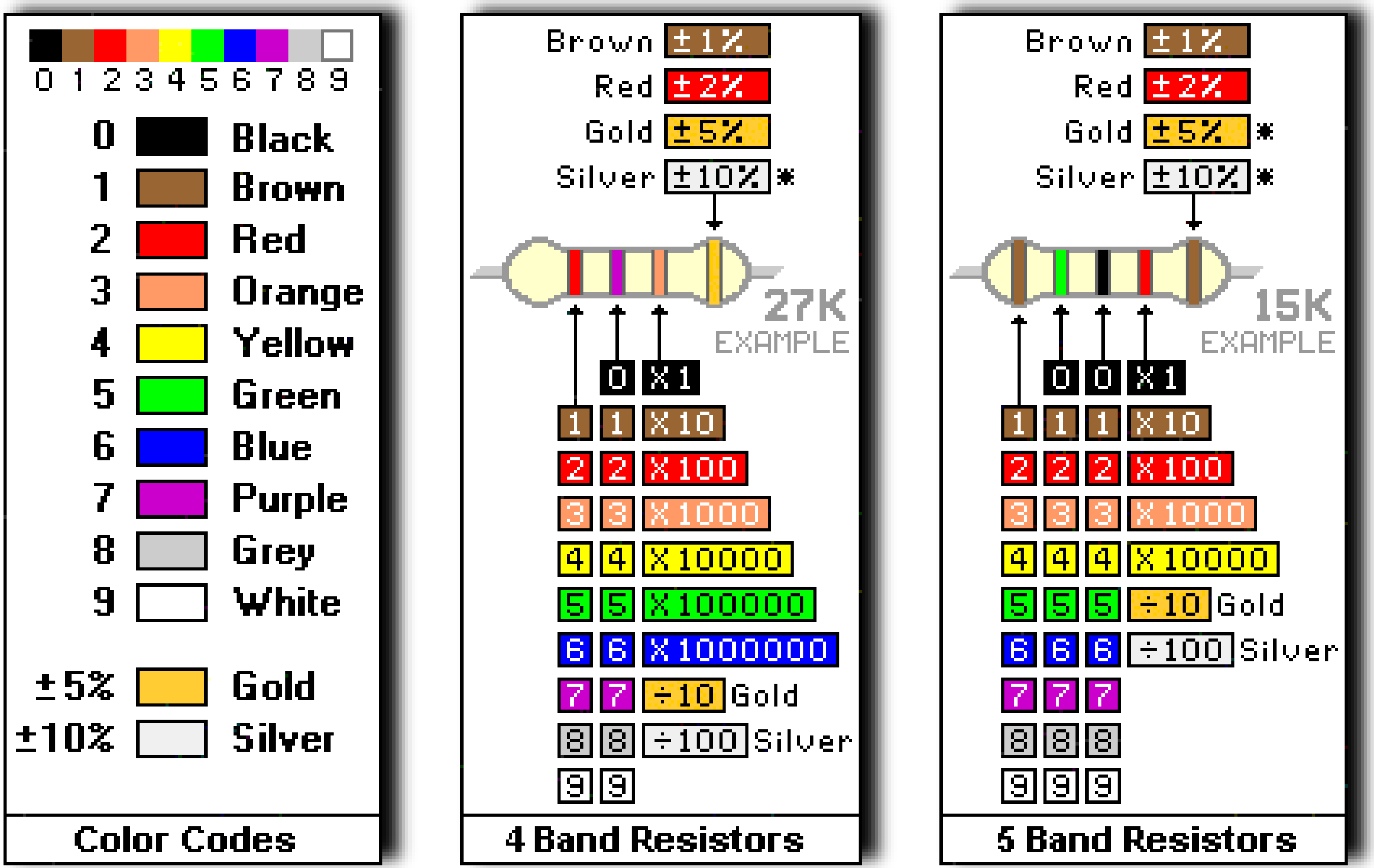
- <u>Stability</u>
  - The value of a resistor will change because of the passage of time and changes in temperature and humidity.
  - Resistor values also change slowly when subject to very high voltage (kV range).
  - In general, these changes are small and can be ignored, except in precision circuits.

- Resistor Imperfections
  - Inductive and capacitive components of a resistor are called parasitic components. These parasitic components depend on the construction techniques used in making the resistor, as well as on the frequency of operation.
  - A real resistor may be represented by pure and ideal elements: a resistor in series with an inductor with a capacitor in parallel with the series combination.
  - Current flowing through a resistor causes electronic noise. The amount and nature of this noise depend on the current, the resistance, and the type of resistor.

- Resistor Marking Schemes
  - Most resistors are small, colored cylinders with wires sticking out of each end of the cylinder. The value (two digits of the resistance value plus a power-of-ten multiplier) and tolerance can be determined from the colored bands on the resistor.
  - The digits 0 through 9 are represented by the following colors:

Black 0	Orange 3	Violet 7
Brown 1	Yellow 4	Gray 8
Red 2	Green 5	White 9
	Blue 6	

  - Similarly, the multiplier power of ten (0 through 8) is given by the same color code.
  - Gold represents a 0.1 multiplier and a ± 5% tolerance.
  - Silver represents a 0.01 multiplier and a ± 10% tolerance.

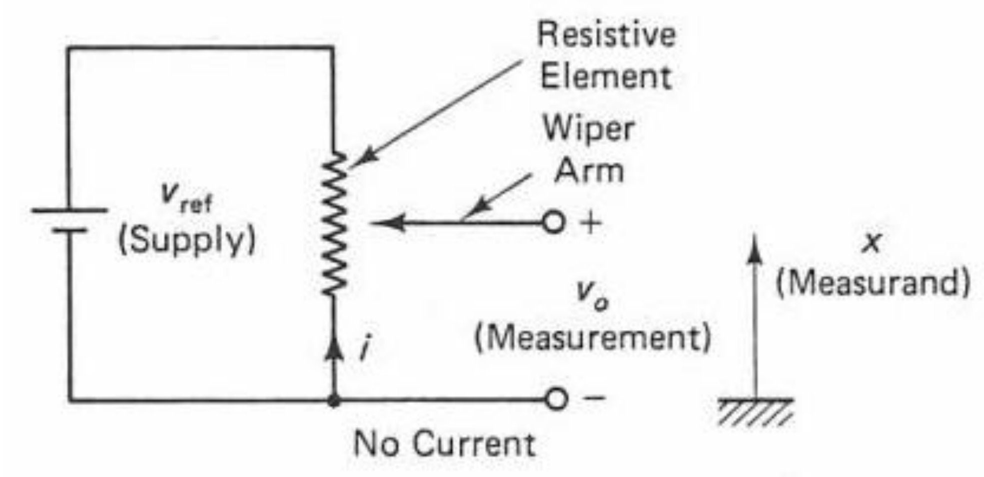

0 1 2 3 4 5 6 7 8 9
0 Black
1 Brown
2 Red
3 Orange
4 Yellow
5 Green
6 Blue
7 Purple
8 Grey
9 White
±5% Gold
±10% Silver
Color Codes
Brown ±1%
Red ±2%
Gold ±5%
Silver ±10% *
27K EXAMPLE
0 0 X1
1 1 X10
2 2 X100
3 3 X1000
4 4 X10000
5 5 X100000
6 6 X1000000
7 7 ÷10 Gold
8 8 ÷100 Silver
9 9
4 Band Resistors
Brown ±1%
Red ±2%
Gold ±5% *
Silver ±10% *
15K EXAMPLE
0 0 0 X1
1 1 1 X10
2 2 2 X100
3 3 3 X1000
4 4 4 X10000
5 5 5 ÷10 Gold
6 6 6 ÷100 Silver
7 7 7
8 8 8
9 9 9
5 Band Resistors

- Fixed Resistor Types and Characteristics
  - Carbon Composition
    - Low in cost and readily available
    - Available in low-accuracy values 20%, 10%, and 5%, resistance values 10Ω to 22MΩ, and power ratings ¼ to 5 W
    - Relatively poor thermal stability and sensitive to humidity
  - Carbon Film
    - Workhorses of modern equipment
    - Compared to carbon composition resistors: better performance, more stable, better noise characteristics, insensitive to humidity
    - Good stability under high voltage; fair thermal stability
    - Available in same resistance and power ranges as carbon composition resistors; 5% is normal tolerance
    - Preferred resistor for non-critical applications

- Metal Film or Metal Oxide
  - Preferred resistor for critical circuitry
  - Much better characteristics than previous two types; better thermal stability and lower noise characteristics
  - Available only in low-power values (¼ W or less) and semi-precision (1% or 2%) or high-precision (0.5% to 0.01%) tolerances
  - More expensive than previous two types
- Wire Wound
  - Generally used in high-power equipment; power ratings run from 1 to 1500 W
  - Available in resistance range of 0.1 to 100 kΩ
  - Generally available in 5% to 10% tolerance range
  - Good noise and stability characteristics
  - Generally quite expensive and large

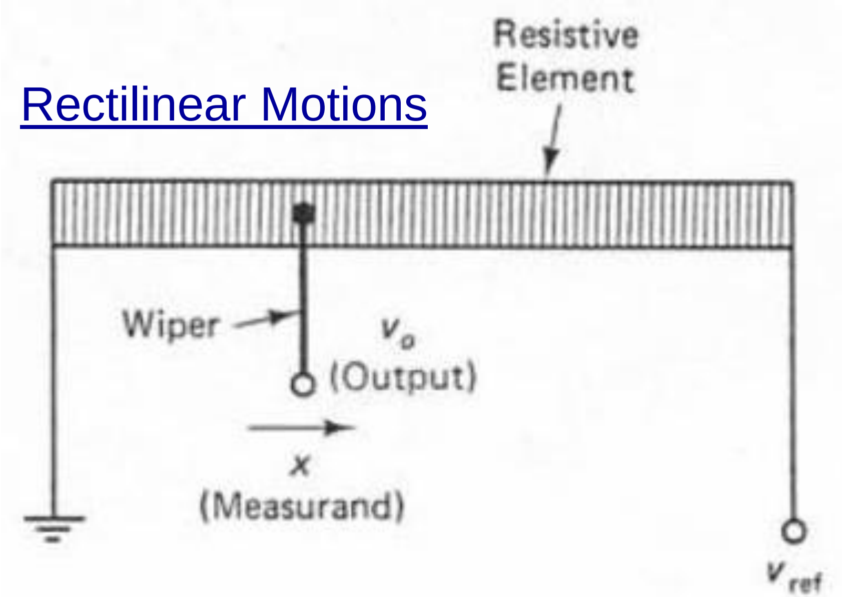
Potentiometer

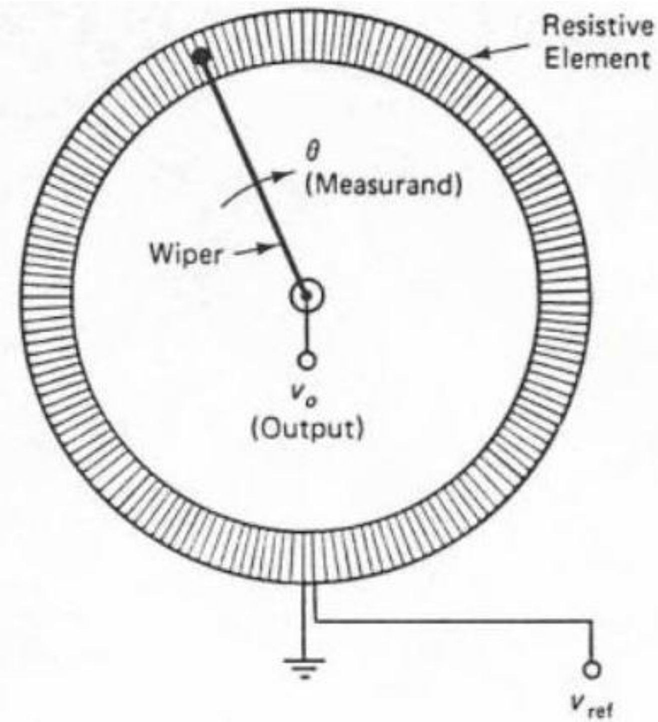
Schematic Diagram



Practical Potentiometer Sensor Configurations

Rectilinear Motions





Angular Motions

- Potentiometer
  - A potentiometer consists of a uniform coil of wire or a film of high-resistive material (e.g., carbon, platinum, conductive plastic) whose resistance is proportional to its length.
  - It has two uses: as a variable resistance and as a displacement sensor.
  - Displacement Sensor
    - A fixed voltage v_{ref} is applied across the coil or film using an external, constant DC voltage supply.
    - The transducer output signal v_o is the DC voltage between the moving contact (wiper arm) sliding on the coil and one terminal of the coil.

$v_o = kx$

- Slider displacement is proportional to the output voltage, which assumes that the output terminals are open-circuit.
- When we assume that the output terminals are open-circuit, we are assuming an infinite-impedance load (or resistance in the present DC case) present at the output terminals, so that the output current is zero.
- In actual practice, the load (the circuitry into which the potentiometer is fed) has a finite impedance and so the output current (through the load) is nonzero.
- The output voltage thus drops, even if the reference voltage v_{ref} is assumed to remain constant under load variations (i.e., the voltage source has zero output impedance).
- This consequence is known as the *loading effect* of the sensor and the linear relationship is no longer valid. An error in the displacement reading results.

Kirchhoff's Circuit Laws

- We focus on basic analysis techniques which apply to all electrical circuits.
- Just as Newton's Law is basic to the analysis of mechanical systems, so are Kirchhoff's Laws basic to electrical circuits.
- One needs to know how to use these laws and combine this with knowledge of the current/voltage behavior of the basic circuit elements to analyze a circuit model.
  - This is a very important point. Kirchhoff's Laws always apply. To use them, however, we need to add the constitutive equations for the elements in the circuit, e.g., e = iR for a resistor.

- Kirchhoff's Voltage (Loop) Law (KVL)
  - It is merely a statement of an intuitive truth; it requires no mathematical or physical proof.
  - <u>KVL says that the algebraic sum of the voltages around any closed circuit path is zero</u>.
  - This law can also be stated in alternative forms:
    - The summation of voltage drops around a closed loop must be zero at every instant.
    - The summation of voltage rises around a closed loop must be zero at every instant.
    - The summation of the voltage drops around a closed loop must equal the summation of the voltage rises at every instant.

- Kirchhoff's Current (Node) Law (KCL)
  - It is based on the physical fact that at any point (node) in a circuit there can be no accumulation of electric charge. In circuit diagrams we connect elements (R, L, C, etc.) with wires which are considered perfect conductors.
  - KCL says that the algebraic sum of currents flowing into or out of a junction is zero.
  - This law can also be stated in alternative forms:
    - The summation of currents into a node must be zero.
    - The summation of currents out of a node must be zero.
    - The summation of currents into a node must equal the summation of currents out.

- In mechanical systems we need sign conventions for forces and motions; in electrical systems we need them for voltages and currents.
- If the assumed positive direction of a current has not been specified at the beginning of a problem, an orderly analysis is quite impossible.
- For voltages, the sign conventions consist of + and – signs at the terminals where the voltage exists.
- Once sign conventions for all the voltages and currents have been chosen, combination of Kirchhoff's Laws with the known voltage/current relations which describe the circuit elements leads us directly to the system differential equations.

KVL and KCL Application Examples

- There are 4 example problems worked on the next two slides.
- Problem #2 shows the application of the Superposition Principle.
  - Here there is a current source and a voltage source. The system is a linear one. Superposition – which says that the output due to several inputs applied simultaneously is equal to the sum of the outputs with each input applied separately – applies here. So we apply the voltage source, replace the current source with an open circuit, and find the output. Then we apply the current source, replace the voltage source with a short circuit, and find the output. The total output with both sources applied will be equal to the sum of the two outputs.
- Problem #4 shows two methods of analysis. The first assigns separate currents to each resistor. The second method – the loop current method – applies a current to each loop.

KVL, KCL, Ohm's Law Example Problems

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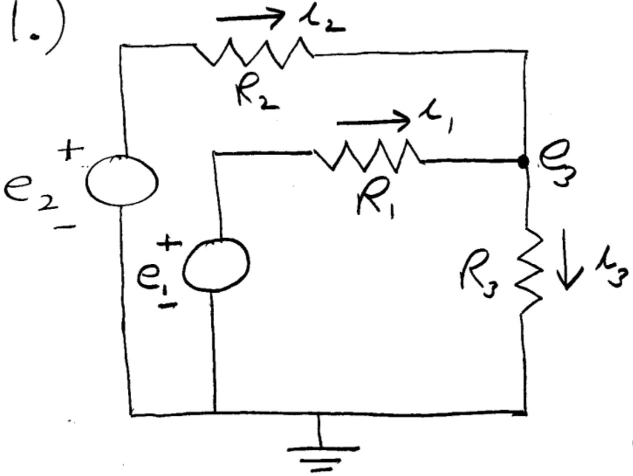
1.)



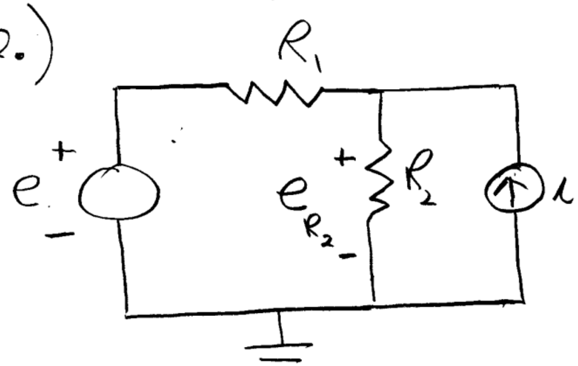
Derive an expression for e_3 as a function of e_1 and e_2.

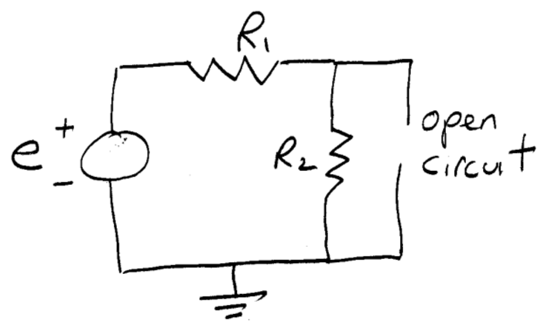
KCL $\quad i_2 + i_1 = i_3$

$$\frac{e_2 - e_3}{R_2} + \frac{e_1 - e_3}{R_1} = \frac{e_3}{R_3}$$

$$e_3 = \left[\frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}\right]\left[\frac{e_1}{R_1} + \frac{e_2}{R_2}\right]$$

2.)





Voltage Divider

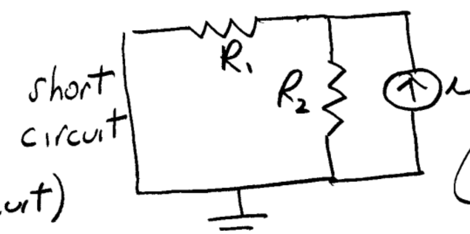
$$e_{R_2} = e\left(\frac{R_2}{R_1 + R_2}\right)$$

what is e_{R_2}?

Use superposition:

e (replace i with open circuit)

i (replace e with short circuit)

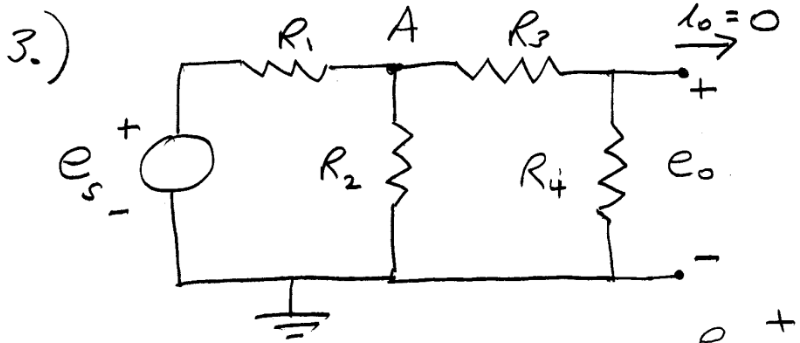


Current Divider

$$e_{R_2} = R_2\left(i\,\frac{R_1}{R_1 + R_2}\right)$$

$\left(e_{R_2}\right)_{total}$ = sum of the two

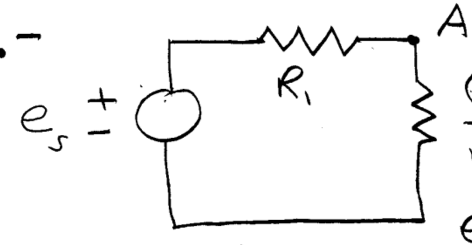
3.)



Find e_o.

$$e_o = \frac{R_4}{R_3+R_4}\, e_A$$

$$(R_3+R_4) \parallel R_2 \Rightarrow \frac{(R_3+R_4)(R_2)}{R_3+R_4+R_2}$$

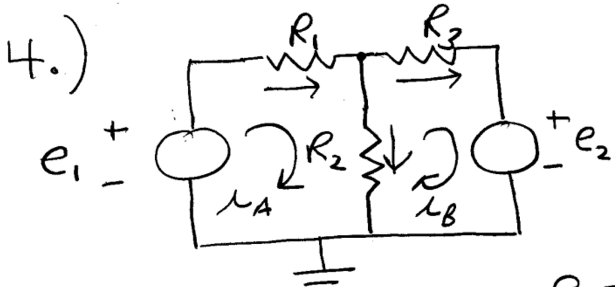


$$\frac{(R_3+R_4)(R_2)}{R_3+R_4+R_2} = R_{eq}$$

$$e_A = \frac{R_{eq}}{R_{eq}+R_1}\, e_s$$

Therefore $e_o = \left(\frac{R_4}{R_3+R_4}\right)\left(\frac{R_{eq}}{R_{eq}+R_1}\right) e_s$

4.)



Find i_{R_1} i_{R_2} i_{R_3}.

<u>KVL</u> using currents through each resistor:

$e_1 - i_{R_1}R_1 - i_{R_2}R_2 = 0$

$e_2 + i_{R_3}R_3 - i_{R_2}R_2 = 0$

<u>KCL</u>

$i_{R_1} = i_{R_2} + i_{R_3}$

$e_1 - i_{R_1}R_1 - i_{R_3}R_3 - e_2 = 0$ outer loop

(not independent – difference of two KVL eqs.)

3 equations – 3 unknowns.

<u>OR</u> Use Loop Currents.

KVL

$$\left.\begin{aligned} &e_1 - i_A R_1 - (i_A - i_B)R_2 = 0 \\ &e_2 + i_B R_3 - (i_A - i_B)R_2 = 0 \end{aligned}\right\} \begin{gathered}\text{2 equations} \\ \text{2 unknowns} \\ i_A \quad i_B\end{gathered}$$

$$\begin{cases} i_{R_1} = i_A \\ i_{R_3} = i_B \\ i_{R_2} = i_A - i_B \end{cases}$$

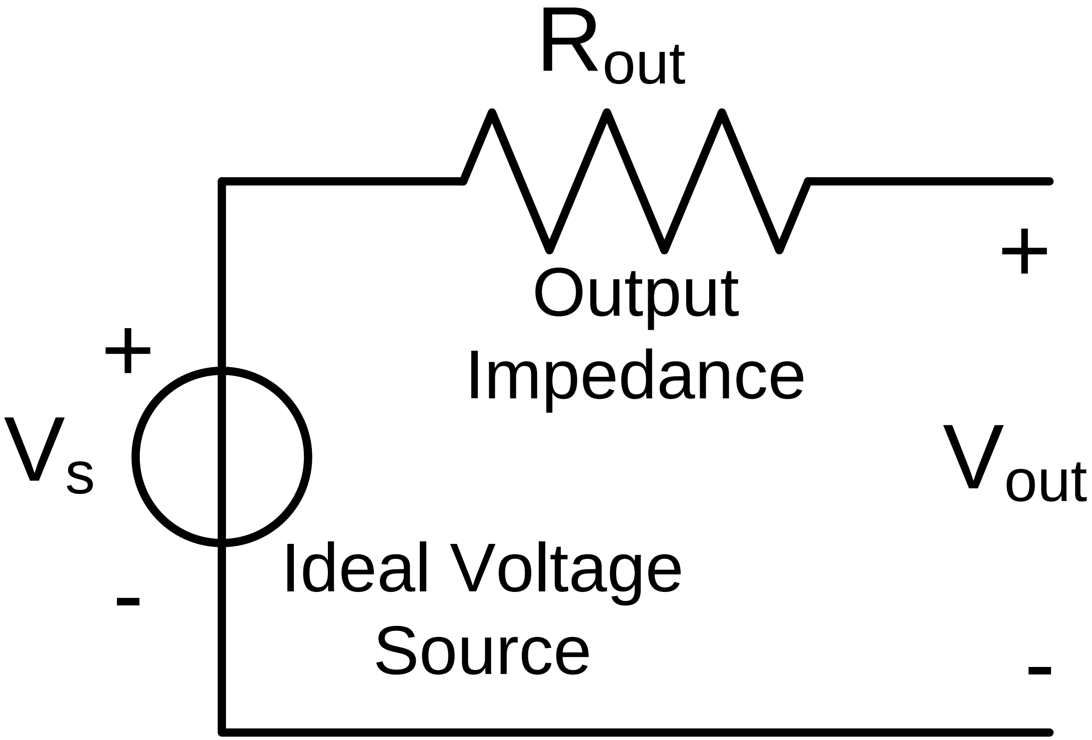
Current and Voltage Sources & Meters: Ideal and Real

- Ideal Voltage Source
  - Supplies the intended voltage to the circuit no matter how much current (and thus power) this might require
  - Can supply infinite current
  - Zero output impedance, i.e., internal resistance is zero.
- Ideal Current Source
  - Supplies the intended current to the circuit no matter how much voltage (and thus power) this might require
  - Can supply infinite voltage
  - Infinite output impedance, i.e., internal resistance is infinite.

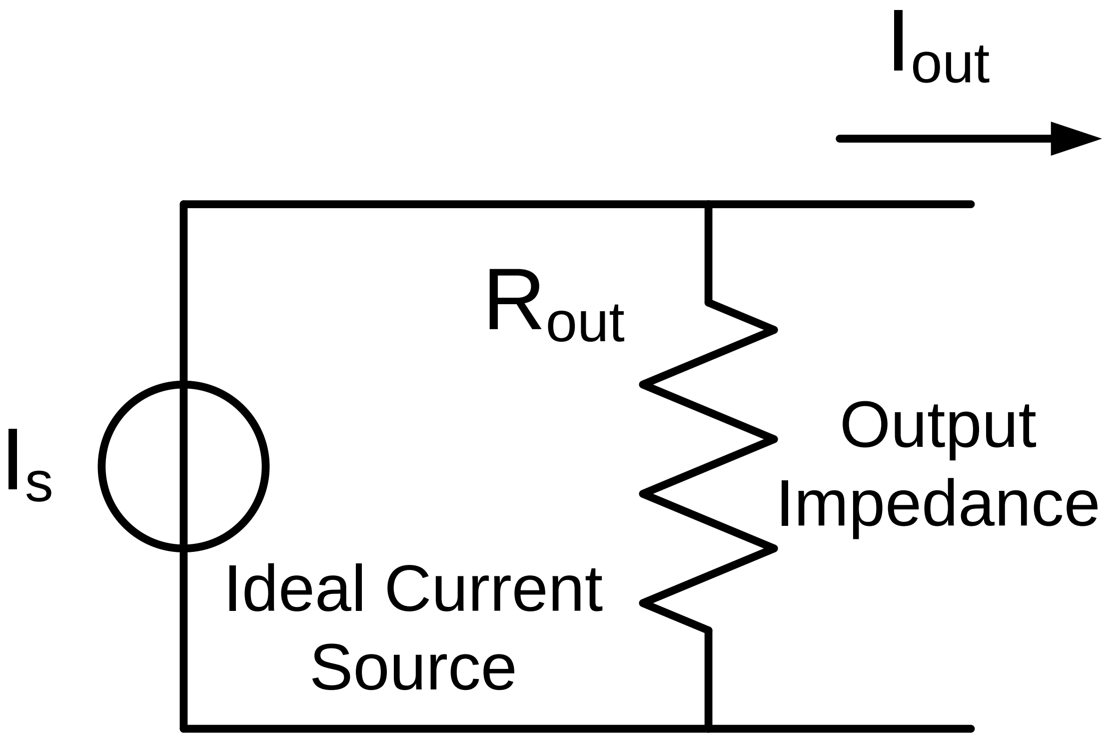
- Real sources have terminal characteristics that are somewhat different from the ideal cases.
- However, the terminal characteristics of the real sources can be modeled using ideal sources with their associated input and output resistances.
- Real Voltage Source
  - Modeled as an ideal voltage source in series with a resistance called the output impedance of the device.
  - When a load is attached to the source and current flows, the output voltage V_{out} will be different from the ideal voltage source V_s due to voltage division.
  - The output impedance (i.e., the internal resistance) of most voltage sources is usually very small (fraction of an ohm).

- For most applications, the output impedance is small enough to be neglected. However, the output impedance can be important when driving a circuit with small resistance because the impedance adds to the resistance of the circuit.

- Real Current Source
  - Modeled as an ideal current source in parallel with an output impedance.
  - When a load is attached to the source, the source current I_s divides between the output impedance and the load.
  - The output impedance (i.e., internal resistance) of most current sources is usually very large, minimizing the current division effect.
  - However, the impedance can be important when driving a circuit with a large resistance.

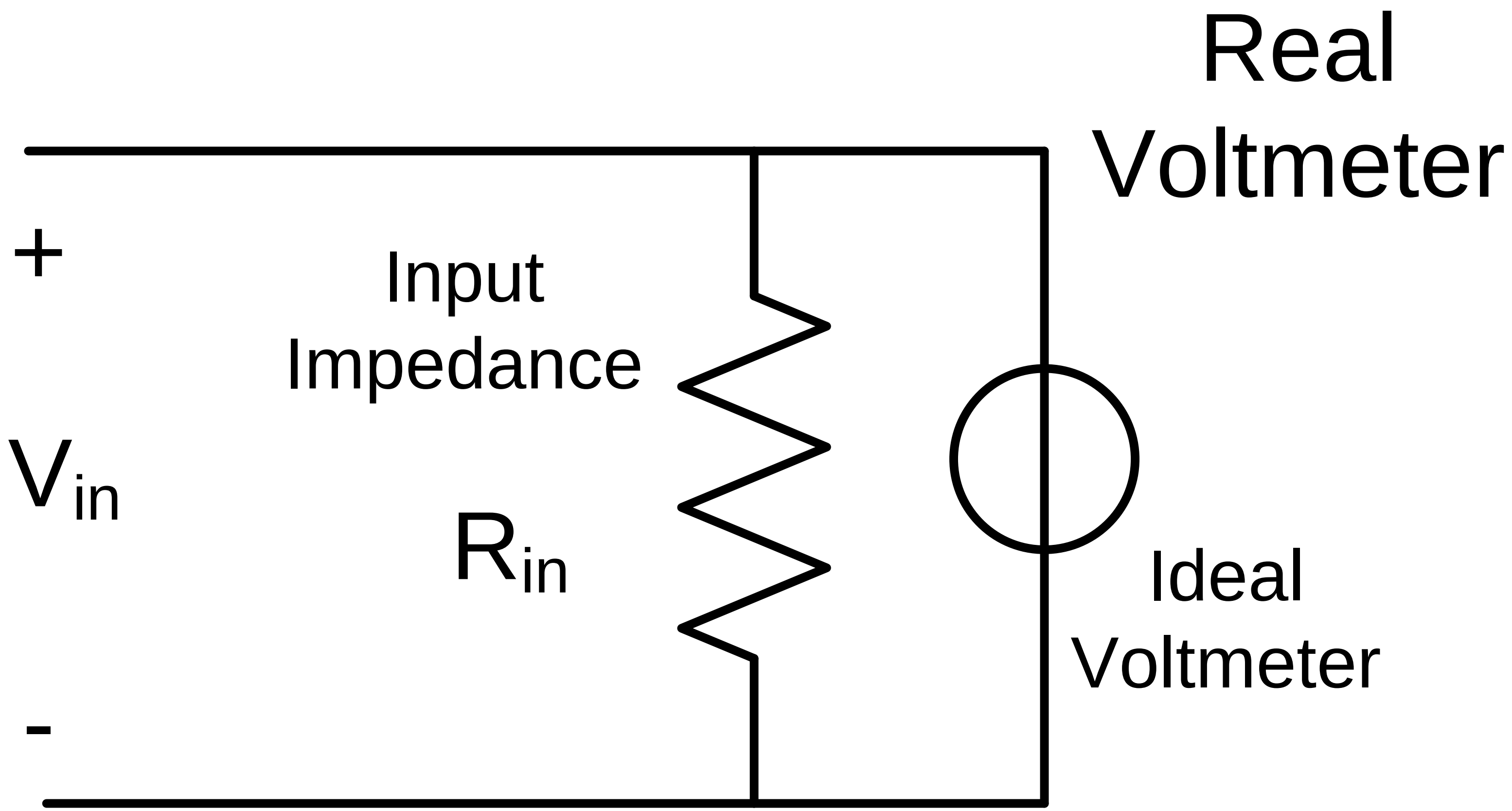


Real Voltage
Source

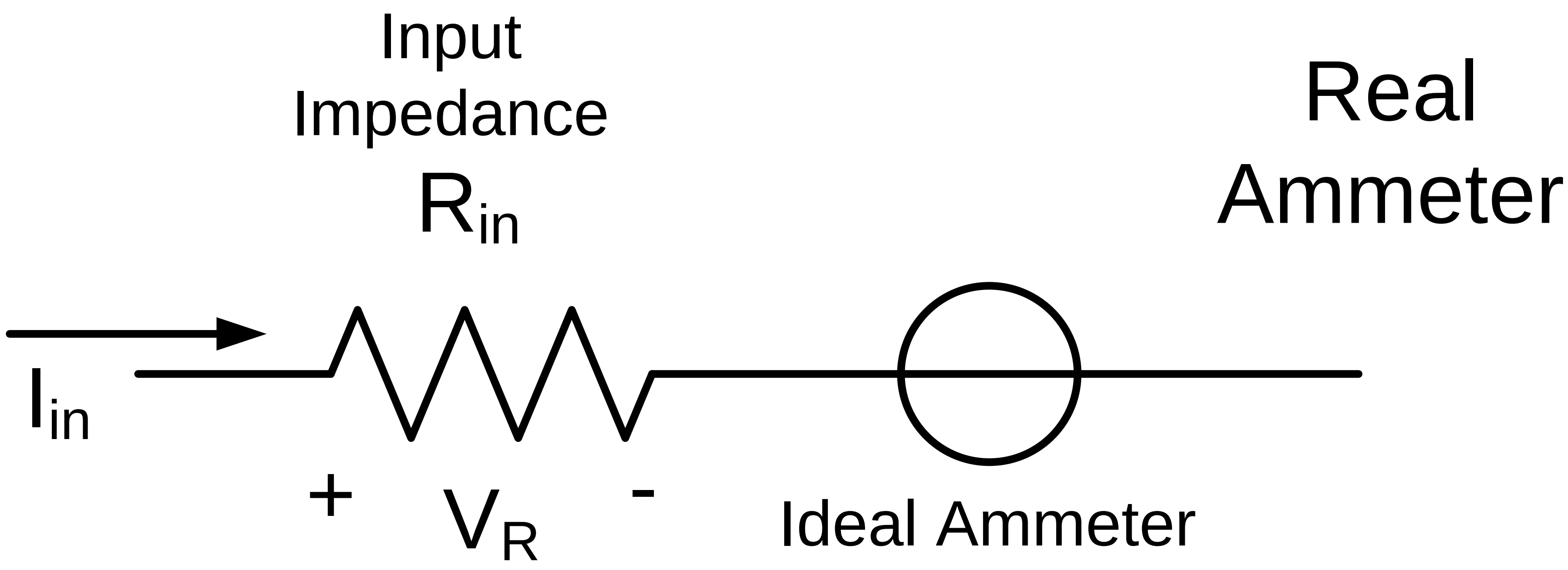


Real Current
Source

- Ideal Voltmeter
  - Infinite input impedance
  - Draws no current
- Real Voltmeter
  - Can be modeled as an ideal voltmeter in parallel with an input impedance.
  - The input impedance is usually very large (1 to 10 MΩ).
  - However, this resistance must be considered when making a voltage measurement across a circuit branch with large resistance since the parallel combination of the meter input impedance and the circuit branch would result in significant error in the measured value.


Real
Voltmeter
+
Input
Impedance
V_{in}
R_{in}
Ideal
Voltmeter
-

- Ideal Ammeter
  - Zero input impedance
  - No voltage drop across it
- Real Ammeter
  - Can be modeled as an ideal ammeter in series with a resistance called the input impedance of the device.
  - The input impedance is usually very small, minimizing the voltage drop V_R added in the circuit.
  - However, this resistance can be important when making a current measurement through a circuit branch with small resistance because the output impedance adds to the resistance of the branch.


Input
Impedance
R_{in}
Real
Ammeter
I_{in}
+
V_R
-
Ideal Ammeter

Function Generator + Voltage Divider + Oscilloscope

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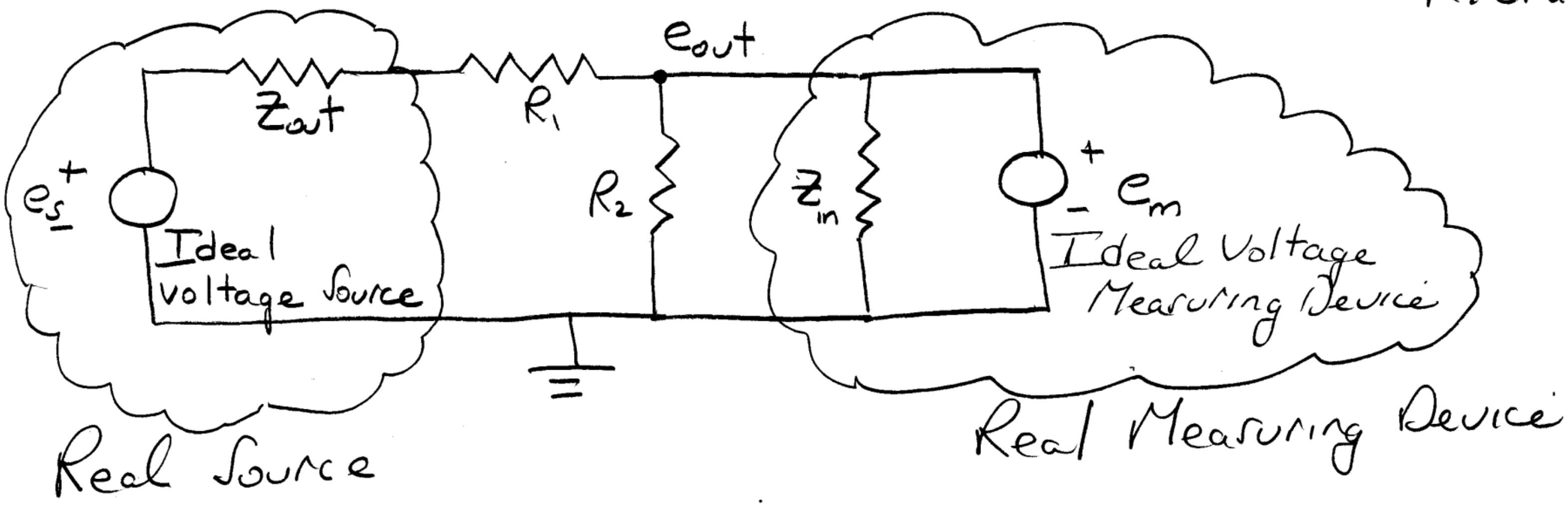


$Z_{out} = 50\,\Omega$ (Function Generator)

$Z_{in} = 1\,M\Omega$ (Oscilloscope)

Ideally

$$e_{out_{Ideal}} = e_s \frac{R_2}{R_1 + R_2}$$

For Real System

$$e_{out} = \frac{\dfrac{R_2 Z_{in}}{R_2 + Z_{in}}}{Z_{out} + R_1 + \dfrac{R_2 Z_{in}}{R_2 + Z_{in}}}$$

$R_2 \parallel Z_{in}$ and R_1 in series with Z_{out}

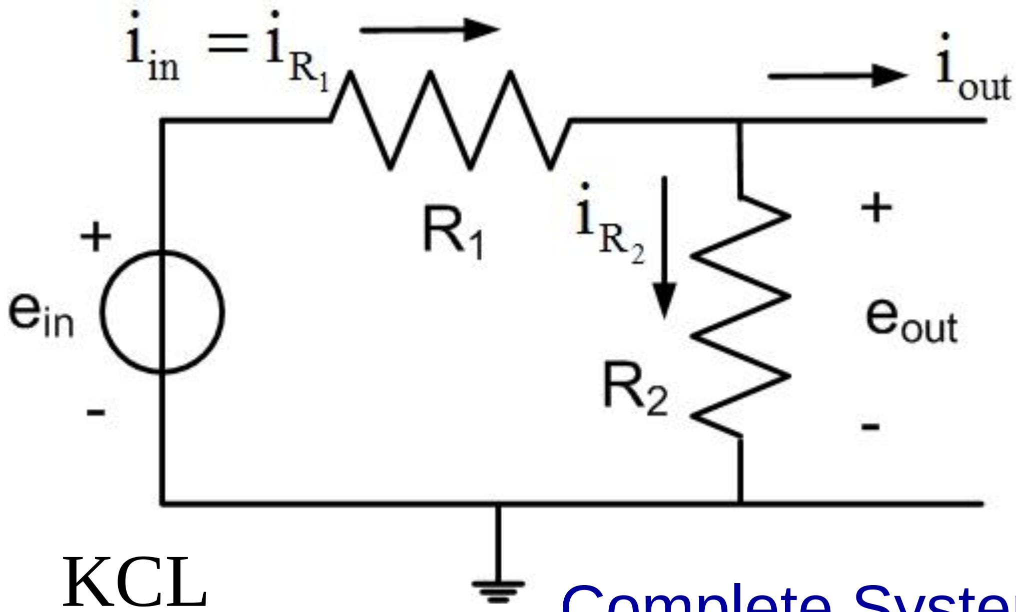
If $R_2 \ll Z_{in}$ and Z_{out} is small $\Leftarrow$

$$e_{out_{Real}} = \frac{\dfrac{R_2}{1 + R_2/Z_{in}}}{Z_{out} + R_1 + \dfrac{R_2}{1 + R_2/Z_{in}}}$$

then $e_{out_{Real}} \approx e_{out_{ideal}}$

Ideal: $Z_{out} = 0$, $Z_{in} = \infty$

Circuit Loading



$$\dot{i}_{in} = \dot{i}_{out} + \dot{i}_{R_2} \quad \text{KCL}$$

$$\dot{i}_{R_2} = \frac{e_{out}}{R_2} \quad \text{Ohm's Law}$$

$$e_{in} - \dot{i}_{in}R_1 - e_{out} = 0 \quad \text{KVL}$$

<u>Complete System Description</u>

$$\begin{bmatrix} e_{out} \\ \dot{i}_{out} \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ \dfrac{-1}{R_2} & \dfrac{R_2 + R_1}{R_2} \end{bmatrix} \begin{bmatrix} e_{in} \\ \dot{i}_{in} \end{bmatrix}$$

$$\begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ \dfrac{-1}{R_2} & \dfrac{R_2 + R_1}{R_2} \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

Complete System Description

$$e_{out} = e_{in} - R_1 i_{in}$$

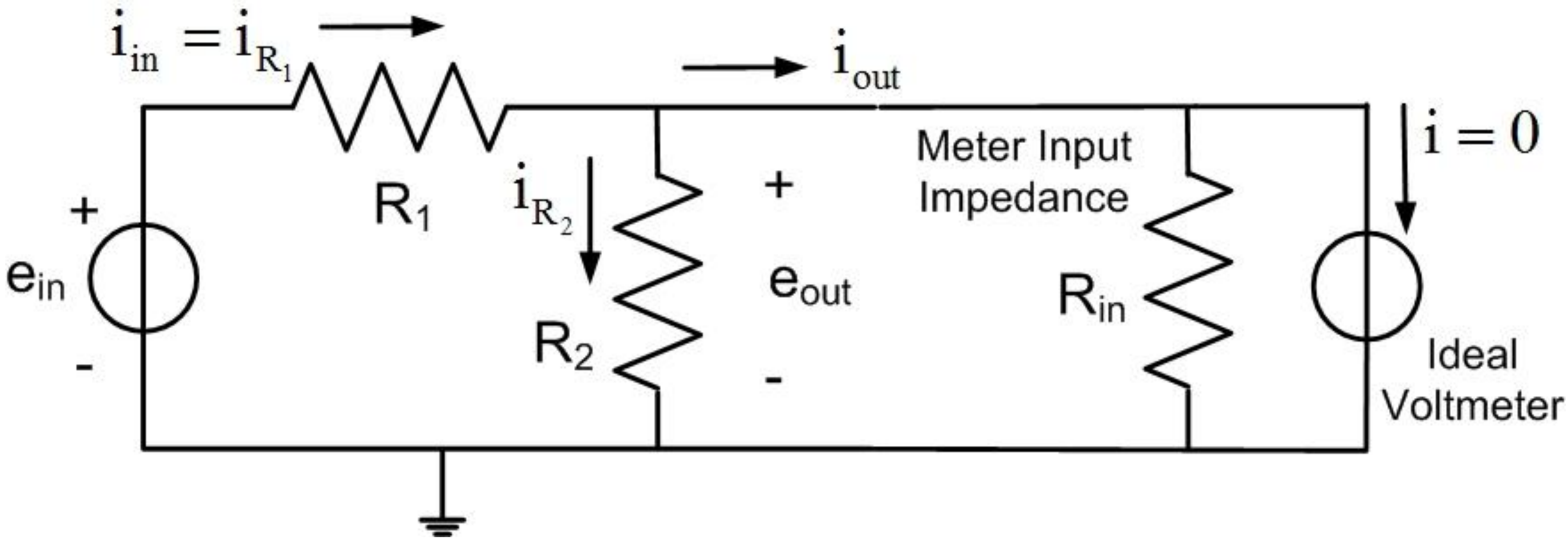
Ideal (No-Loading) Case: $i_{out} = 0$

$$0 = \frac{-e_{in}}{R_2} + \frac{R_2 + R_1}{R_2} i_{in}$$

$$0 = \frac{-e_{in}}{R_2} + \frac{R_2 + R_1}{R_2} i_{in} \Rightarrow i_{in} = \frac{e_{in}}{R_2 + R_1}$$

$$\begin{aligned} e_{out} &= e_{in} - R_1 i_{in} \\ &= e_{in} - R_1 \left[\frac{e_{in}}{R_2 + R_1} \right] = e_{in} \left[\frac{R_2}{R_2 + R_1} \right] \end{aligned}$$

Ideal Voltage Divider

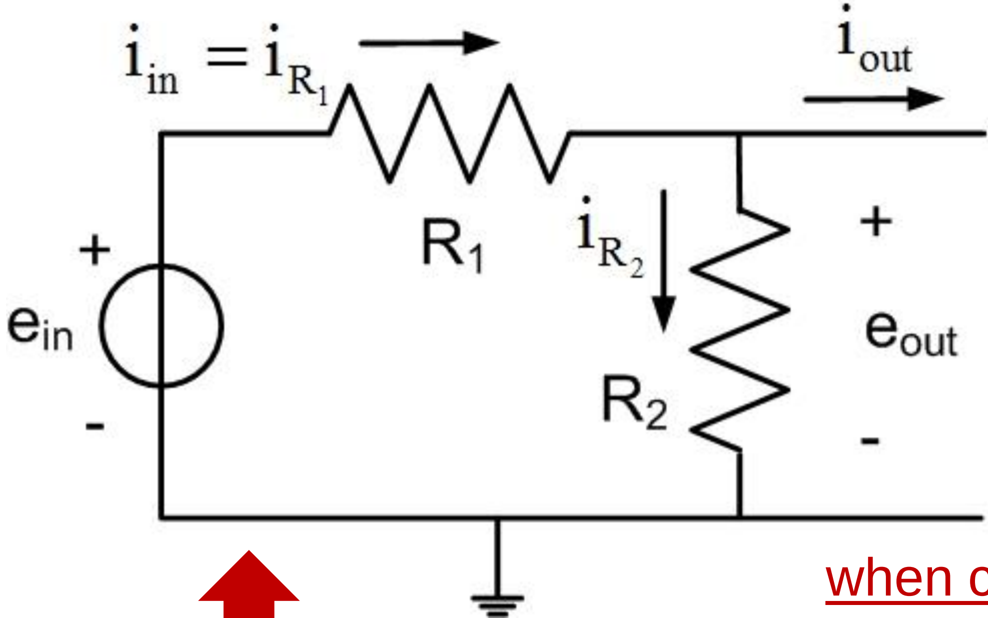


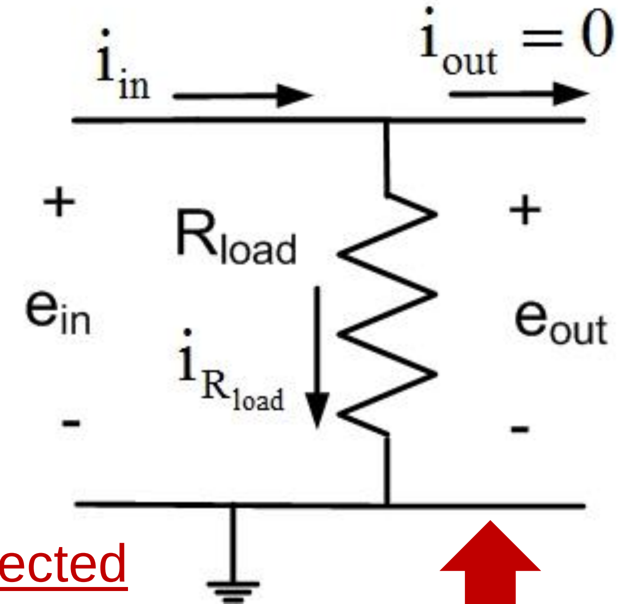
This is a voltage divider: R_1 and $R_2 \parallel R_{in}$.

$$e_{out} = e_{in} \frac{\left(\dfrac{R_2 R_{in}}{R_2 + R_{in}}\right)}{R_1 + \left(\dfrac{R_2 R_{in}}{R_2 + R_{in}}\right)} = e_{in} \frac{\left(\dfrac{R_2}{(R_2 / R_{in}) + 1}\right)}{R_1 + \left(\dfrac{R_2}{(R_2 / R_{in}) + 1}\right)}$$

If $R_2 << R_{in}$ No Loading

Meter $R_{in} \approx 10\ M\Omega$ Scope $R_{in} \approx 1\ M\Omega$





<u>when connected</u>

$$i_{out} = i_{in}$$

$$e_{out} = e_{in}$$

$$\begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ \dfrac{-1}{R_2} & \dfrac{R_2 + R_1}{R_2} \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

$$\begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{-1}{R_{load}} & 1 \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

General Loading Case

$$\begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{-1}{R_{load}} & 1 \end{bmatrix} \begin{bmatrix} 1 & -R_1 \\ \dfrac{-1}{R_2} & \dfrac{R_2 + R_1}{R_2} \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -R_1 \\ \dfrac{-1}{R_{load}} + \dfrac{-1}{R_2} & \dfrac{R_1}{R_{load}} + \dfrac{R_2 + R_1}{R_2} \end{bmatrix} \begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix}$$

$$e_{out} = e_{in} - R_1 i_{in}$$

$$i_{out} = \left(\frac{-1}{R_{load}} + \frac{-1}{R_2} \right) e_{in} + \left(\frac{R_1}{R_{load}} + \frac{R_2 + R_1}{R_2} \right) i_{in} = 0$$

Solve for i_{in} in second equation and substitute back into the first equation.

$$i_{in} = \frac{R_{load} + R_2}{R_1 R_2 + R_{load} R_2 + R_1 R_{load}} e_{in}$$

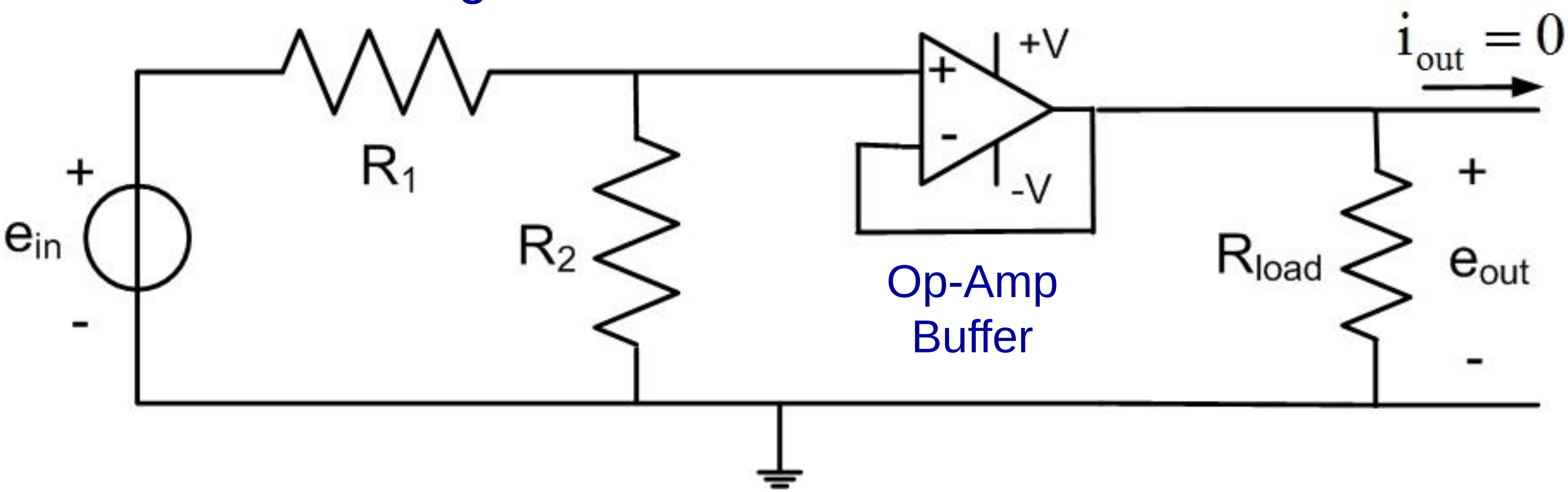
$$e_{out} = e_{in} - R_1 \left(\frac{R_{load} + R_2}{R_1 R_2 + R_{load} R_2 + R_1 R_{load}} e_{in} \right)$$

$$e_{out} = e_{in} \frac{R_2 R_{load}}{R_1 \left(R_2 + R_{load} \right) + R_{load} R_2}$$

$$e_{out} = e_{in} \frac{\dfrac{R_2 R_{load}}{R_2 + R_{load}}}{R_1 + \dfrac{R_{load} R_2}{R_2 + R_{load}}} = e_{in} \frac{\dfrac{R_2}{\dfrac{R_2}{R_{load}} + 1}}{R_1 + \dfrac{R_2}{\dfrac{R_2}{R_{load}} + 1}}$$

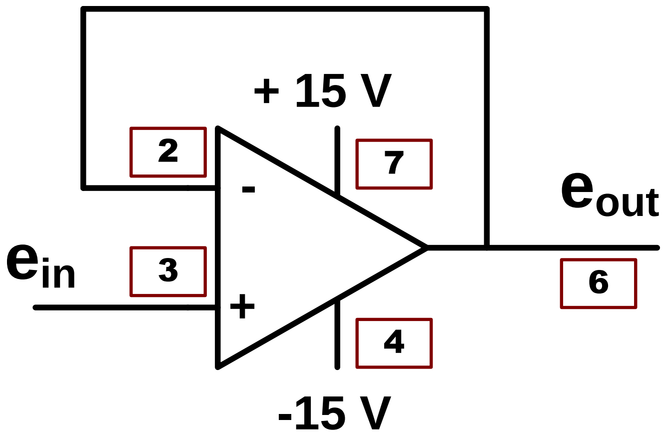
The downstream circuit will always load the upstream circuit? The question is "how much?" What to do?

- Don't ignore it! Know that it is present and model its effect. Adjust parameters accordingly to reduce the effect.
- Or one can prevent the loading by inserting a buffer op-amp in the unity-gain, voltage-follower configuration between the circuits, as shown below. The voltage-divider upstream circuit now behaves in an unloaded way and the load will see the full voltage.



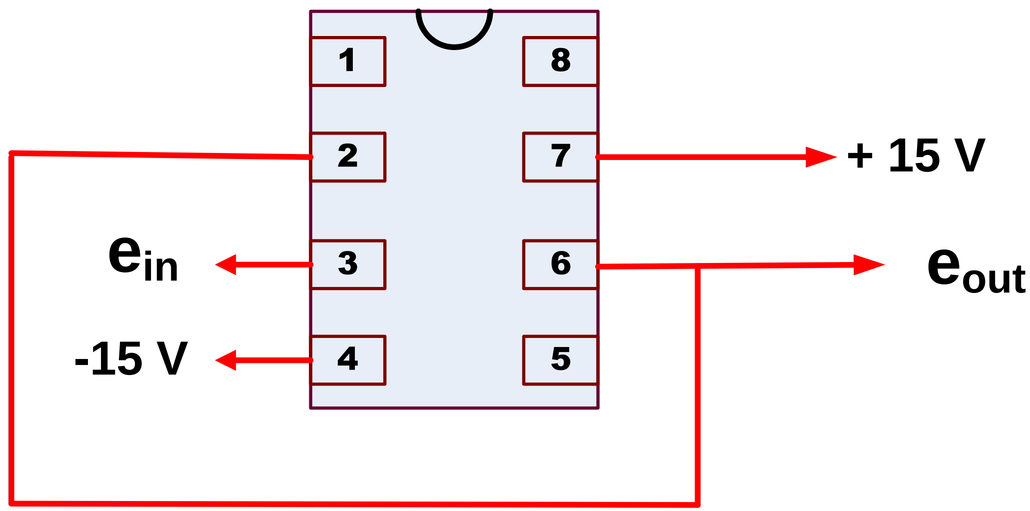
Unity-Gain Buffer Op Amp

$e_{in} = e_{out}$ and in phase



Circuit Diagram Representation

Wiring Diagram



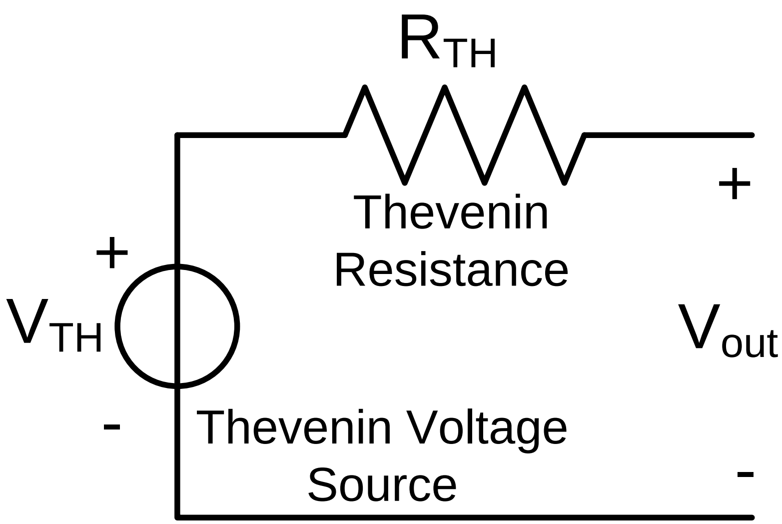
Input Impedance = ∞ Output Impedance = 0

Norton and Thevenin Equivalent Circuits

- Thevenin's Theorem
  - Any circuit network of linear elements can be replaced by an equivalent circuit consisting merely of a single voltage source V_{th} and a single series resistance R_{th}, i.e., a simple real voltage source.
  - Simplifying a complex circuit will simplify the analysis of it.
  - To calculate the values V_{th} and R_{th}, we follow three simple steps:
    - Find the open-circuit (i.e., no current flow) output voltage V_{oc} (across the output terminals without the load being added to the output) of the actual circuit.

- In the actual circuit, replace any voltage sources with a short circuit (i.e., an ideal wire with no resistance) and any current sources with an open circuit (no connection, infinite resistance). Redraw the circuit and determine the total resistance R_{total} now appearing across the output terminals.
- Then, $V_{oc} = V_{th}$ and $R_{total} = R_{th}$.

– Note that we can replace the word resistance with impedance. The concept still is valid.

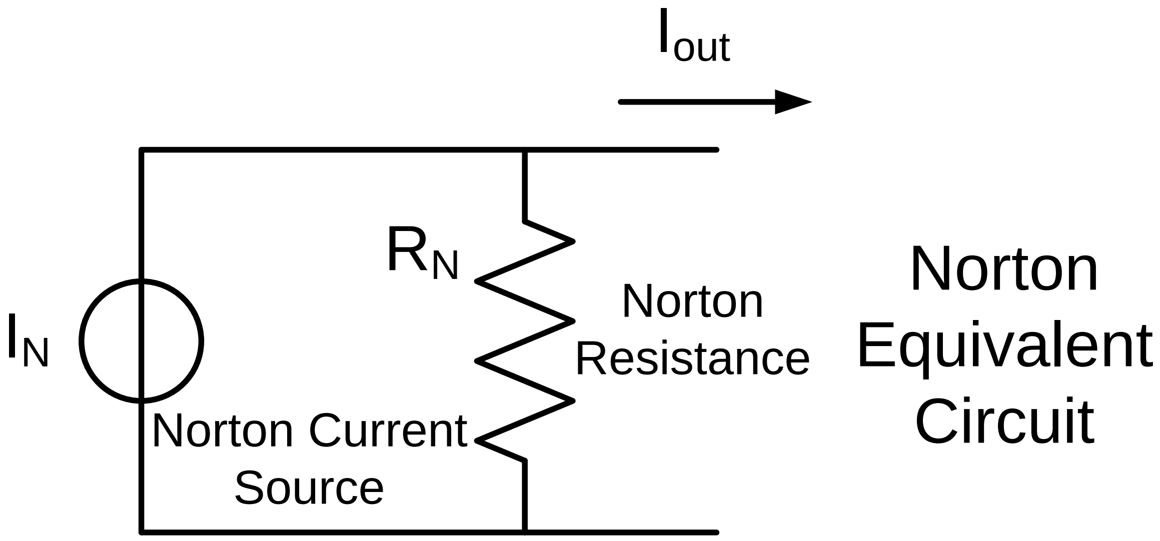


Thevenin Equivalent Circuit

- Norton's Theorem
  - Any complex circuit of linear elements can be simply represented by a single current source I_N and a single parallel resistance R_N.
  - To calculate the values I_N and R_N, we follow three simple steps:
    - Determine the current that would flow through a short circuit connected across the output terminals. This is the short-circuit current flow I_{sc}.
    - In the actual circuit, replace any voltage sources with a short circuit and any current sources with an open circuit. Redraw the circuit and determine the total resistance R_{total} now appearing across the output terminals.

- Then $I_{sc} = I_N$ and $R_{total} = R_N$.

– Note that we can replace the word resistance with impedance. The concept still is valid.

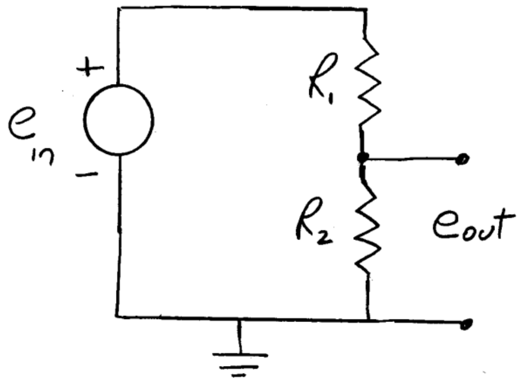


- <u>Relationship between the Thevenin Equivalent Circuit and the Norton Equivalent Circuit</u>
  - A circuit can be simplified by finding either the Thevenin or the Norton equivalents. Which equivalent you choose is dependent only on whether you want the original circuit to act as a voltage source or a current source.
  - The Thevenin and Norton resistances of a circuit are equal, i.e., $R_{th} = R_N$.
  - The voltage developed across the R_N by the I_N is equal to the V_{th}, i.e., $(R_N)(I_N) = V_{th}$.
  - Since $R_N = R_{th}$, we also see that $I_N = V_{th} / R_{th}$.
  - If one of the equivalent circuits has been determined, it is an easy job to find the other.

Equivalent Circuits: Thevenin and Norton

$\boxed{1}$

K. Craig
1-2014



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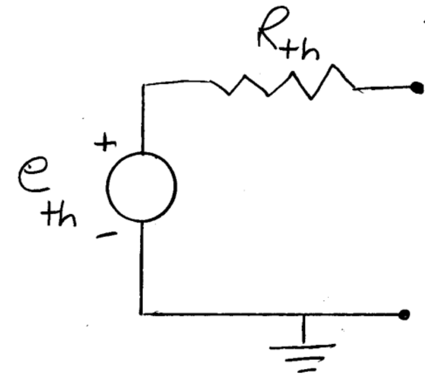
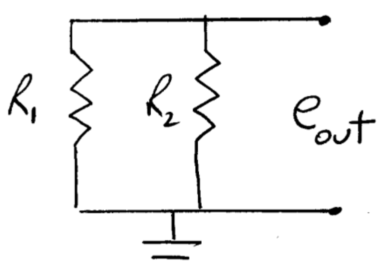


$$\left.\begin{matrix} e_{th} \\ R_{th} \end{matrix}\right\} ?$$

Thevenin Equivalent Circuit

$$e_{oc} = e_{out} = \frac{R_2}{R_1 + R_2} e_{in} = e_{th}$$

Replace e_{in} with a short circuit.



$$R_{Total} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{th} = R_{Total}$$

Summary

$$e_{th} = e_{in} \frac{R_2}{R_1 + R_2}$$

$$R_{th} = \frac{R_1 R_2}{R_1 + R_2}$$

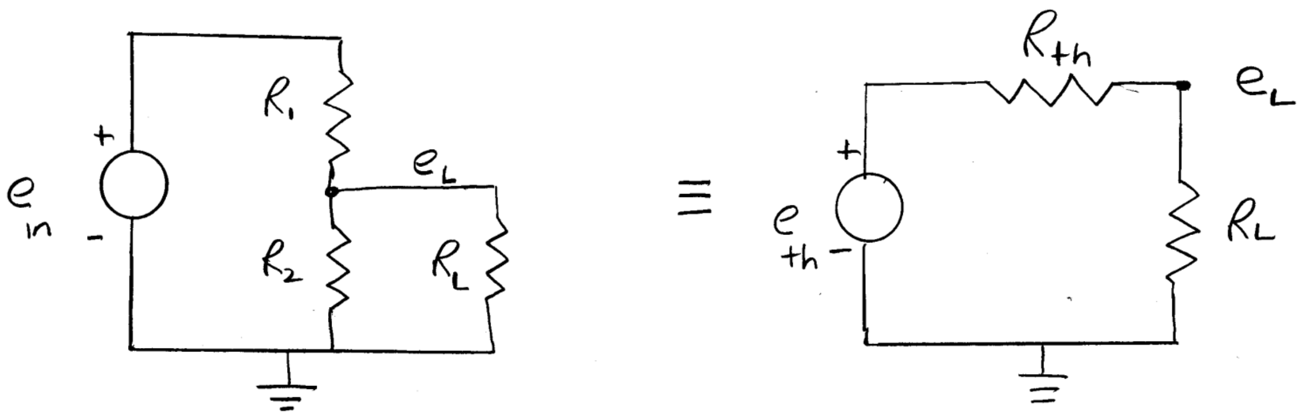
Example: $\begin{cases} R_1 = 18\ k\Omega \\ R_2 = 30\ k\Omega \\ e_{in} = 24V \end{cases}$

$e_{th} = 15V$

$R_{th} = 11.25\ k\Omega$

Why Do This? To Simplify Analysis

2



$R_1 = 18\ \text{k}\Omega$
$R_2 = 30\ \text{k}\Omega$
$e_{in} = 24\text{V}$

$e_{th} = 15\text{V}$
$R_{th} = 11.25\text{V}$

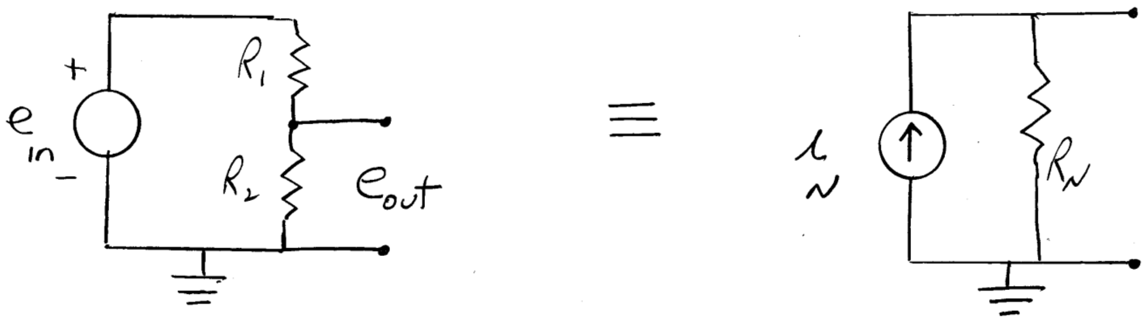
$$e_L = e_{th} \frac{R_L}{R_L + R_{th}}$$

Voltage Divider

Effect of Adding R_L is the Same on Both Circuits.

What is the Norton Equivalent Circuit for the original circuit ?

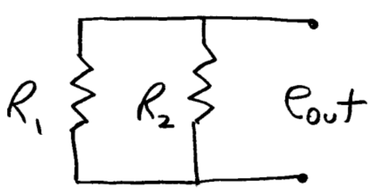
$\boxed{3}$



Norton Equivalent Circuit

$$i_{sc} = e_{in}/R_1 = i_N$$

Replace e_{in} with a short circuit



$$R_{total} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_N = R_{total}$$

But we know $R_N = R_{Th}$!

Summary

$$i_N = e_{in}/R_1$$

$$R_N = \frac{R_1 R_2}{R_1 + R_2}$$

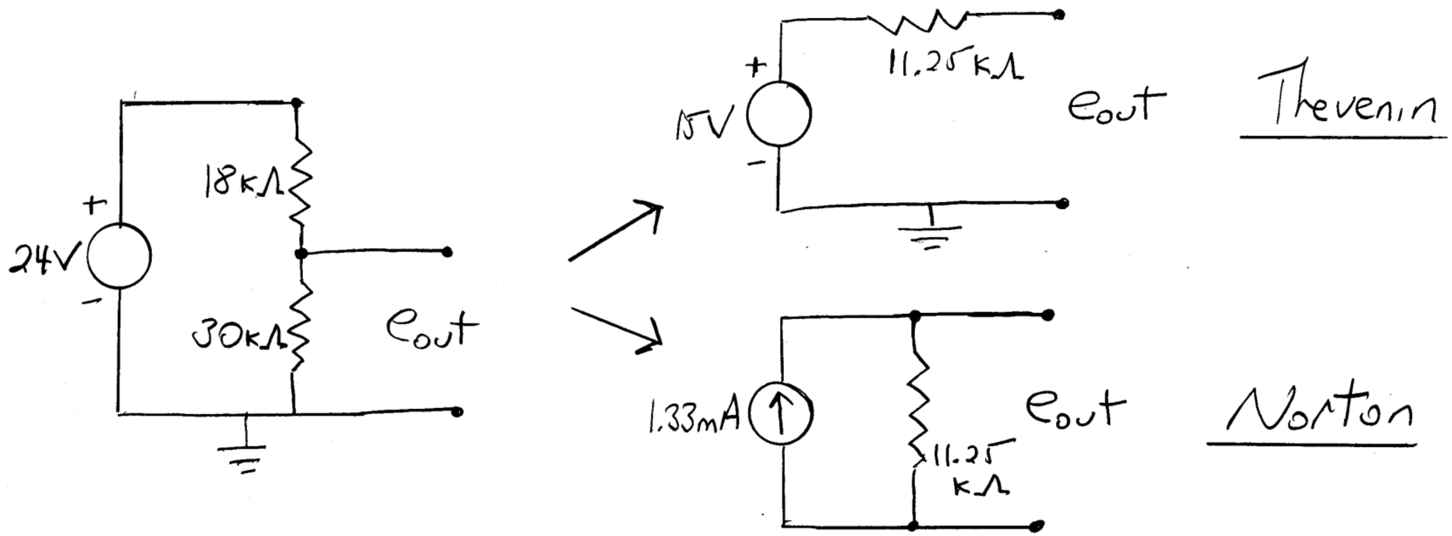
Example:

$$\begin{cases} R_1 = 18\,k\Omega \\ R_2 = 30\,k\Omega \\ e_{in} = 24\,V \end{cases}$$

$i_N = 1.33\,mA$

$R_N = 11.25\,k\Omega$

Relationship between {Norton Equivalent Circuit / Thevenin Equivalent Circuit} $\boxed{4}$



Observations:

1) A circuit can be simplified by finding either the Thevenin or the Norton equivalents.

2) $R_{th} = R_N$

3) $R_N i_N = e_{th}$ $\quad i_N = e_{th}/R_{th}$

{If one equivalent circuit has been determined, the other can be just written down.

Impedance Matching

- The major factor in understanding impedance matching stems from the realization that:
  - The output of almost any circuit can be represented as either a Thevenin Equivalent Circuit or a Norton Equivalent Circuit, where the associated internal impedance is more commonly called the output or source impedance.
  - The load circuit connected to the output of almost any circuit can be represented by a single impedance and is usually called either the load or input impedance.
- Impedance Matching simply means finding the best values for the output impedance of the source and the input impedance of the load to satisfy certain conditions.

- Impedance Matching for Voltage Transfer
  - Voltage transfer means transferring a voltage from a source circuit to a load circuit. We usually want as much voltage as possible to be transferred and, when this has been achieved, we say that we have good voltage transfer.
  - The condition for good voltage transfer is met if $R_S << R_L$. We have acceptable voltage transfer if R_L is at least 10 times larger than R_S.
- Impedance Matching for Current Transfer
  - There are times when we are interested in transferring current rather than voltage. These are when we have a current source (or a circuit which can be represented as a current source) which has a load circuit requiring as much current as possible from the source.

- Current transfer means transferring a current from a source circuit to a load circuit. We usually want as much current as possible to be transferred. When this has been achieved we say that we have good current transfer.
- The condition for good current transfer is met if $R_L << R_S$. We have acceptable current transfer if R_L is at least 10 times smaller than R_S.

- Impedance Matching for Power Transfer
  - Impedance matching for power transfer may be the most important condition for applications involving output to the real world! It tackles the problem of transferring power from one circuit to another.

- For maximum power transfer to occur we need $R_S = R_L$.
- This outcome is not obvious, but it can be deduced intuitively by considering that for maximum output voltage we want $R_S << R_L$ and for maximum output current we want $R_S >> R_L$. But because power = voltage times current, we require a condition which satisfies both inequalities. So it is not surprising that for maximum power transfer $R_S = R_L$.
- Analytically, we can show this as follows:

$$V_L = \frac{R_L}{R_L + R_S} V_S$$

$$P_L = \frac{V_L^2}{R_L} = \frac{R_L}{\left(R_L + R_S\right)^2} V_S^2$$

$$\frac{dP_L}{dR_L} = V_S^2 \frac{\left(R_L + R_S\right)^2 - 2R_L\left(R_L + R_S\right)}{\left(R_L + R_S\right)^4} = 0$$

Solve for R_L: $R_L = R_S$

to maximize power transmission