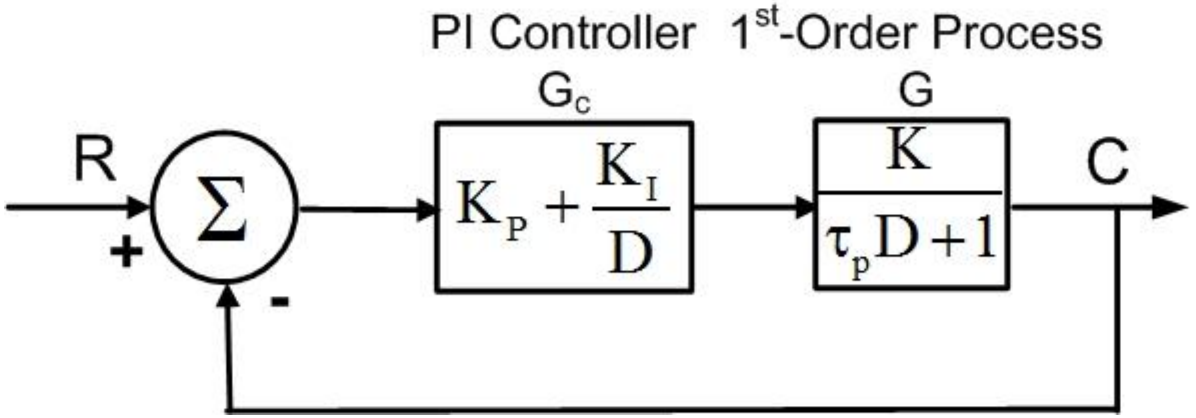
Time Response & Frequency Response of a 2nd-Order Dynamic System



$$\frac{C}{R} = \frac{G_c G}{1 + G_c G} = \frac{\frac{KK_I}{\tau_p}\left(\frac{K_P}{K_I}D + 1\right)}{D^2 + \frac{KK_P + 1}{\tau_p}D + \frac{KK_I}{\tau_p}} = \frac{\omega_n^2(\tau D + 1)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

Closed-Loop Transfer Function: 2nd-Order Dynamic System With Numerator Dynamics

2nd-Order Dynamic System Model

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

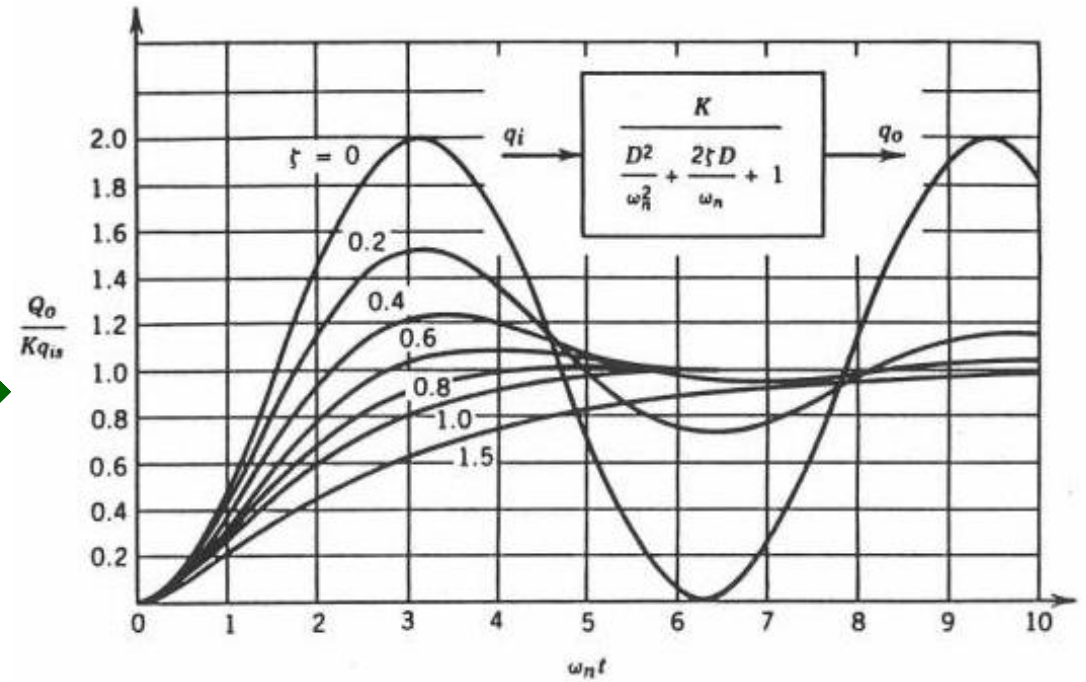
$$\frac{1}{\omega_n^2} \frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_0}{dt} + q_0 = K q_i$$

$$\omega_n \triangleq \sqrt{\frac{a_0}{a_2}} = \text{undamped natural frequency}$$

$$\zeta \triangleq \frac{a_1}{2\sqrt{a_2 a_0}} = \text{damping ratio}$$

$$K \triangleq \frac{b_0}{a_0} = \text{steady-state gain}$$

Step Response
of a
2nd-Order System



$$\frac{1}{\omega_n^2}\frac{d^2q_0}{dt^2}+\frac{2\varsigma}{\omega_n}\zeta\frac{dq_0}{dt}+q_0=Kq_i$$

Step Response of a 2^{nd}-Order System

$$q_o=Kq_{is}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\sin^{-1}\sqrt{1-\zeta^2}\right)\right]\qquad \zeta<1$$

$$q_o=Kq_{is}\left[1-\left(1+\omega_n t\right)e^{-\omega_n t}\right]\qquad \zeta=1$$

$$q_o=Kq_{is}\left[\begin{array}{l}1-\dfrac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}\\ \qquad\qquad +\dfrac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}\end{array}\right]\qquad \zeta>1$$

Frequency Response of a 2^{nd}-Order System

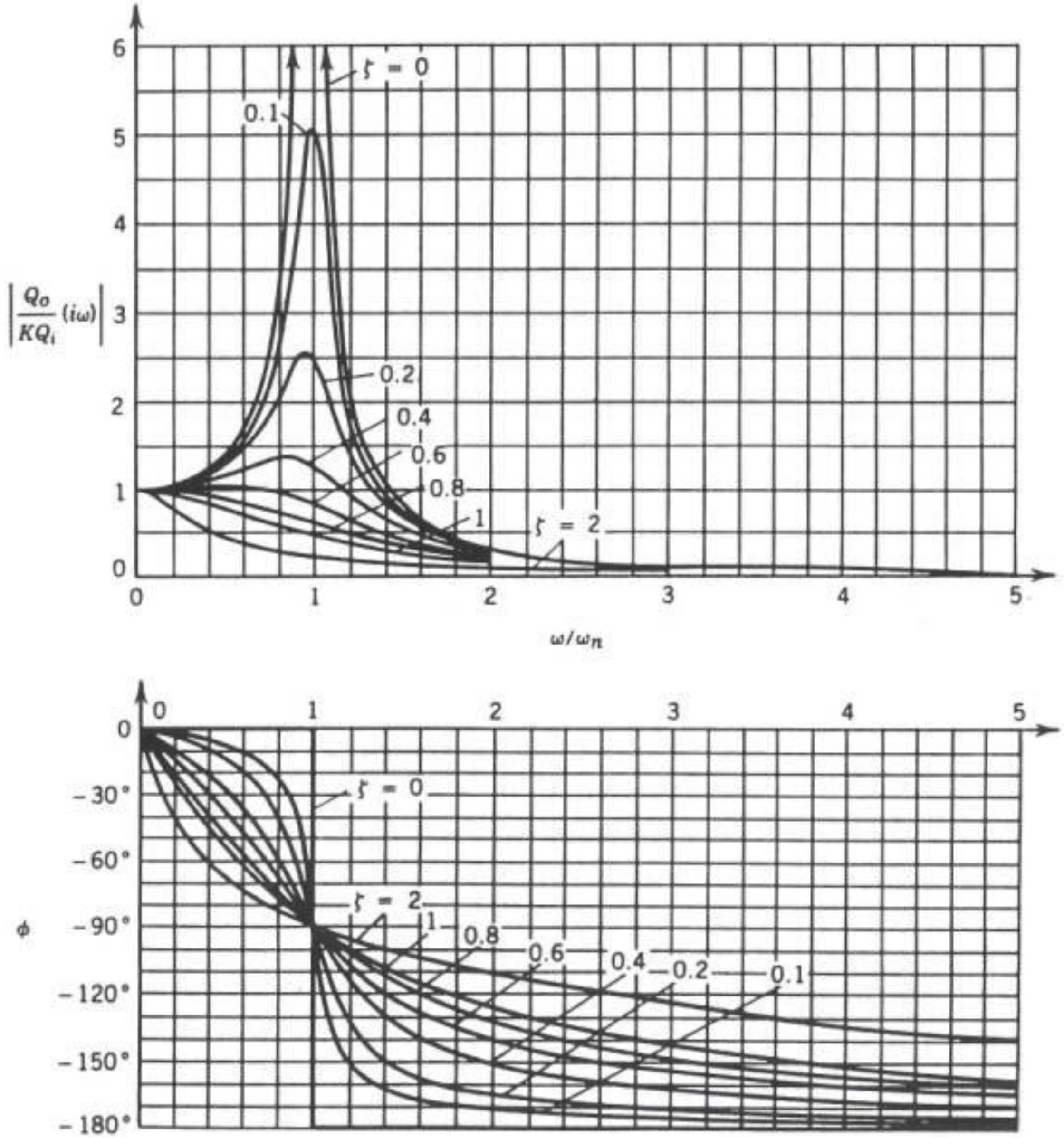
Operational Transfer Function

$$\frac{Q_o}{Q_i}(D) = \frac{K}{\dfrac{D^2}{\omega_n^2} + \dfrac{2\zeta D}{\omega_n} + 1}$$

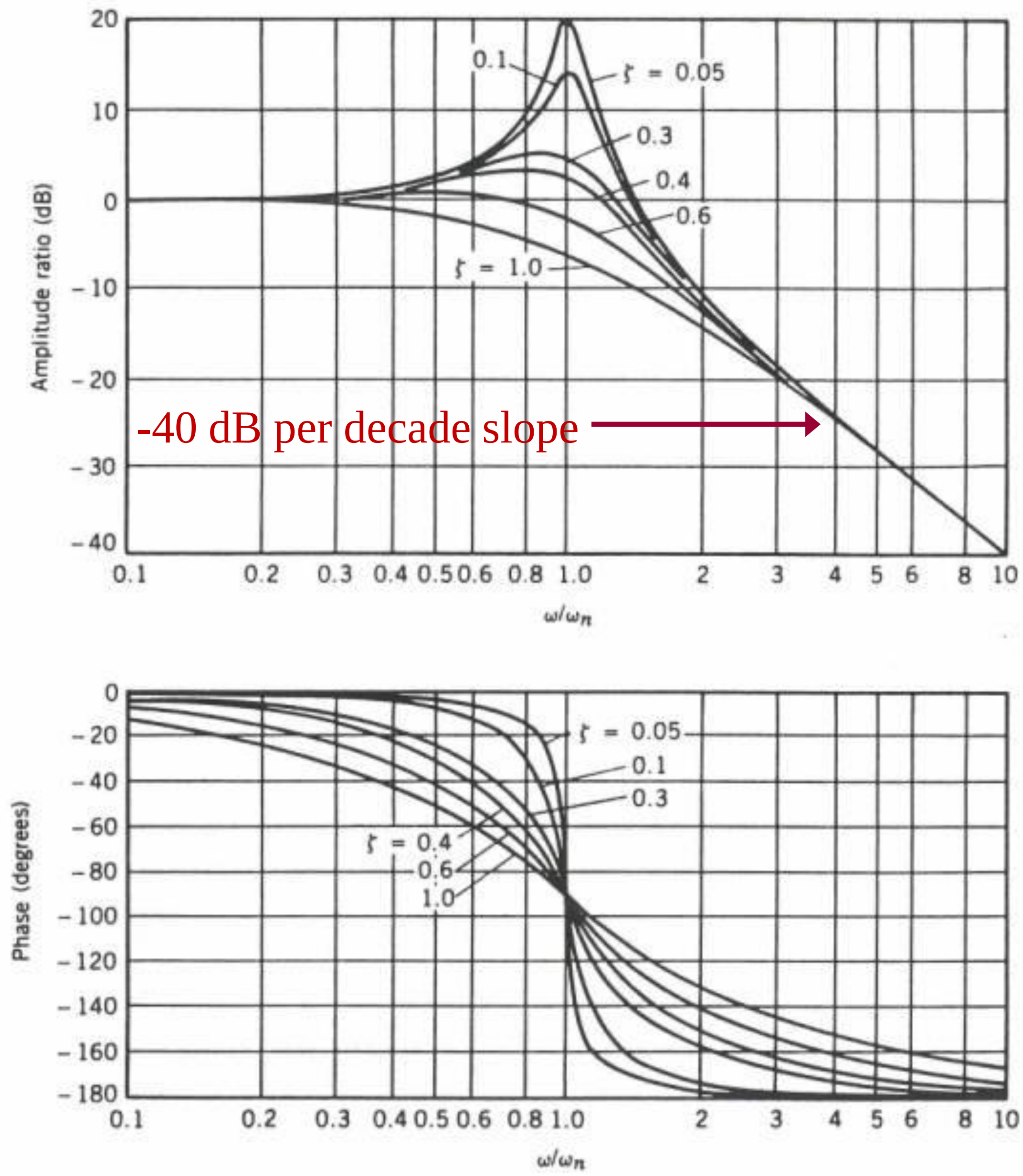
Sinusoidal Transfer Function

$$\frac{Q_o}{Q_i}(i\omega) = \frac{K}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \dfrac{4\zeta^2\omega^2}{\omega_n^2}}} \angle \tan^{-1}\frac{2\zeta}{\left(\dfrac{\omega}{\omega_n} - \dfrac{\omega_n}{\omega}\right)}$$

Frequency Response of a 2^{nd}-Order System



Frequency Response
of a
2nd-Order System



- Some Observations
  - When a physical system exhibits a natural oscillatory behavior, a 1^{st}-order model (or even a cascade of several 1^{st}-order models) cannot provide the desired response. The simplest model that does possess that possibility is the 2^{nd}-order dynamic system model.
  - This system is very important in control design.
    - System specifications are often given assuming that the system is 2^{nd} order.
    - For higher-order systems, we can often use dominant pole techniques to approximate the system with a 2^{nd}-order transfer function.

- Damping ratio ζ clearly controls oscillation; $\zeta < 1$ is required for oscillatory behavior.
- The undamped case ($\zeta = 0$) is not physically realizable (total absence of energy loss effects) but gives us, mathematically, a sustained oscillation at frequency ω_n.
- Natural oscillations of damped systems are at the damped natural frequency ω_d, and not at ω_n.

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- In hardware design, an optimum value of $\zeta = 0.64$ is often used to give maximum response speed without excessive oscillation.
- Undamped natural frequency ω_n is the major factor in response speed. For a given ζ response speed is directly proportional to ω_n.

- Thus, when 2nd-order components are used in feedback system design, large values of ω_n (small lags) are desirable since they allow the use of larger loop gain before stability limits are encountered.
- For frequency response, a resonant peak occurs for ζ < 0.707. The peak frequency is ω_p and the peak amplitude ratio depends only on ζ.

$$\omega_p = \omega_n \sqrt{1-2\zeta^2} \qquad \text{peak amplitude ratio} = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$$

- <u>Bandwidth</u>
  - The bandwidth is the frequency where the amplitude ratio drops by a factor of 0.707 = -3dB of its gain at zero or low-frequency.

- For a 1^{st} -order system, the bandwidth is equal to $1/\tau$.
- The larger (smaller) the bandwidth, the faster (slower) the step response.
- Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.
- For a 2^{nd}-order system:

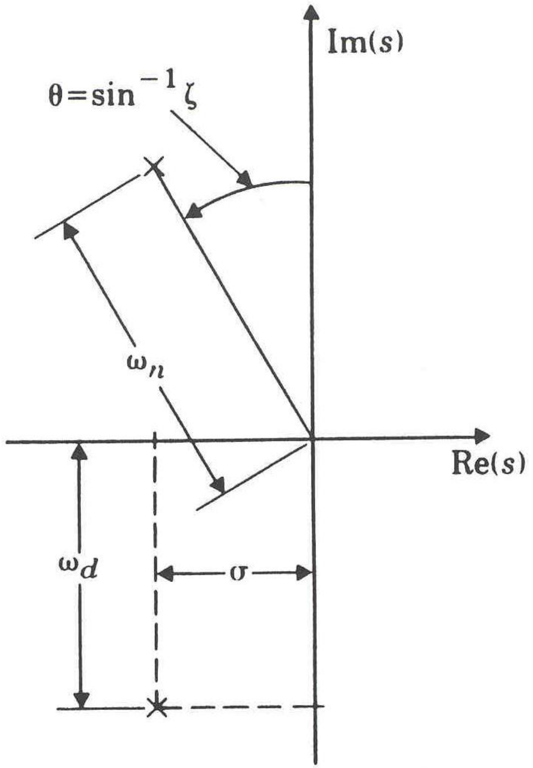
$$BW = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

- As ζ varies from 0 to 1, BW varies from $1.55\omega_n$ to $0.64\omega_n$. For a value of $\zeta = 0.707$, BW = ω_n. For most design considerations, we assume that the bandwidth of a 2^{nd}-order all pole system can be approximated by ω_n.

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\varsigma\omega_n \pm i\omega_n\sqrt{1-\varsigma^2}$$

$$s_{1,2} = -\sigma \pm i\omega_d$$



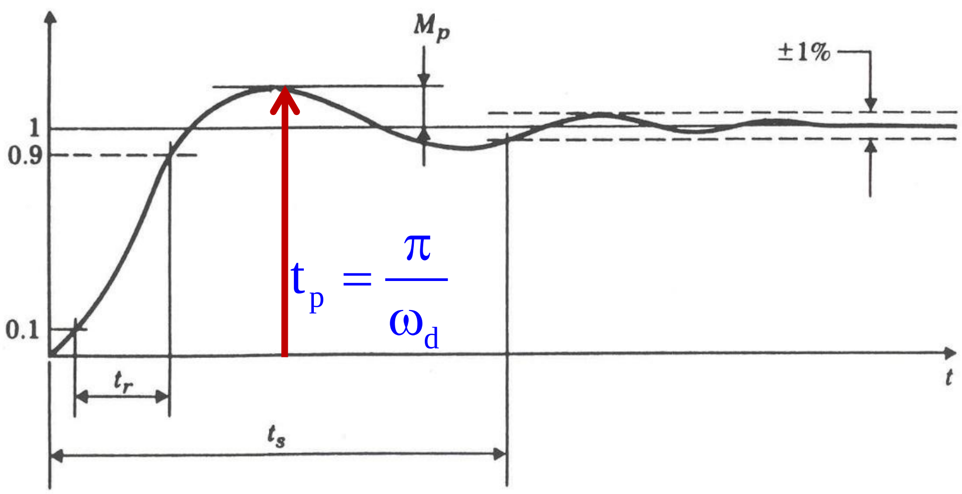
$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$t_r \approx \frac{1.8}{\omega_n} \quad \text{rise time}$$

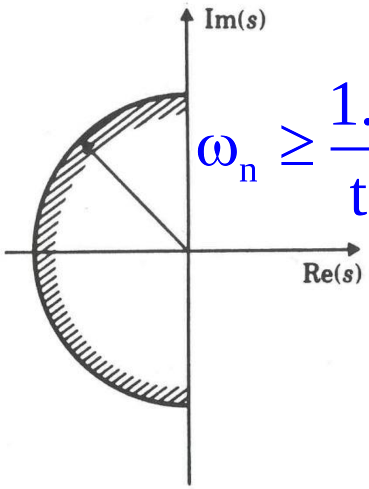
$$t_s \approx \frac{4.6}{\varsigma\omega_n} \quad \text{settling time}$$

$$M_p = e^{\frac{-\pi\varsigma}{\sqrt{1-\varsigma^2}}} \quad (0 \le \zeta < 1) \quad \text{overshoot}$$

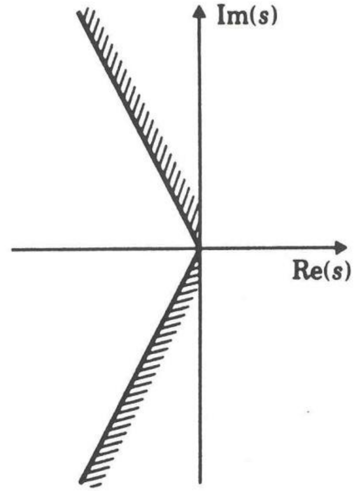
$$\approx \left(1 - \frac{\zeta}{0.6}\right) \qquad (0 \le \zeta \le 0.6)$$

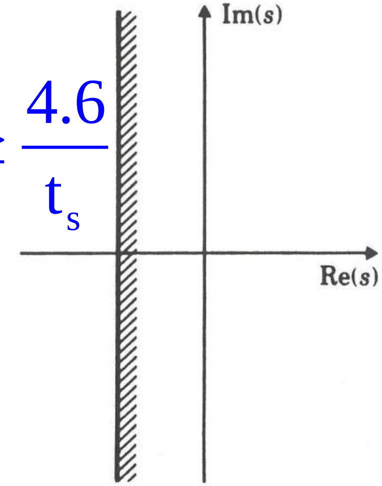


$$t_p = \frac{\pi}{\omega_d}$$



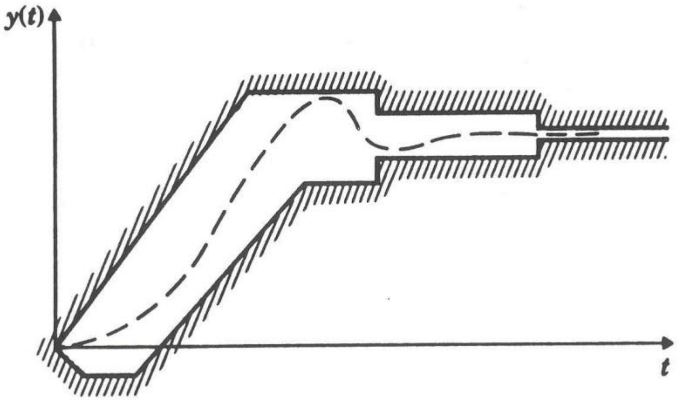
$$\omega_n \geq \frac{1.8}{t_r}$$

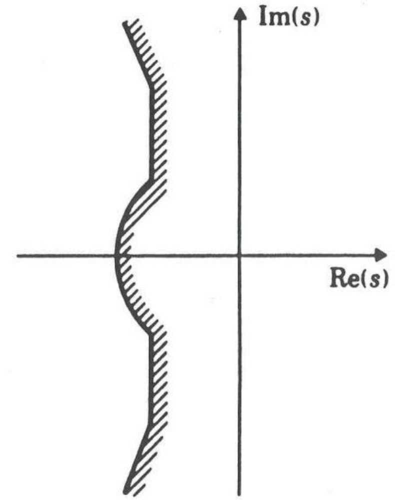




$$\sigma \geq \frac{4.6}{t_s}$$

$$\zeta \geq 0.6\left(1 - M_p\right) \qquad 0 \leq \zeta \leq 0.6$$





Time-Response Specifications vs. Pole-Location Specifications

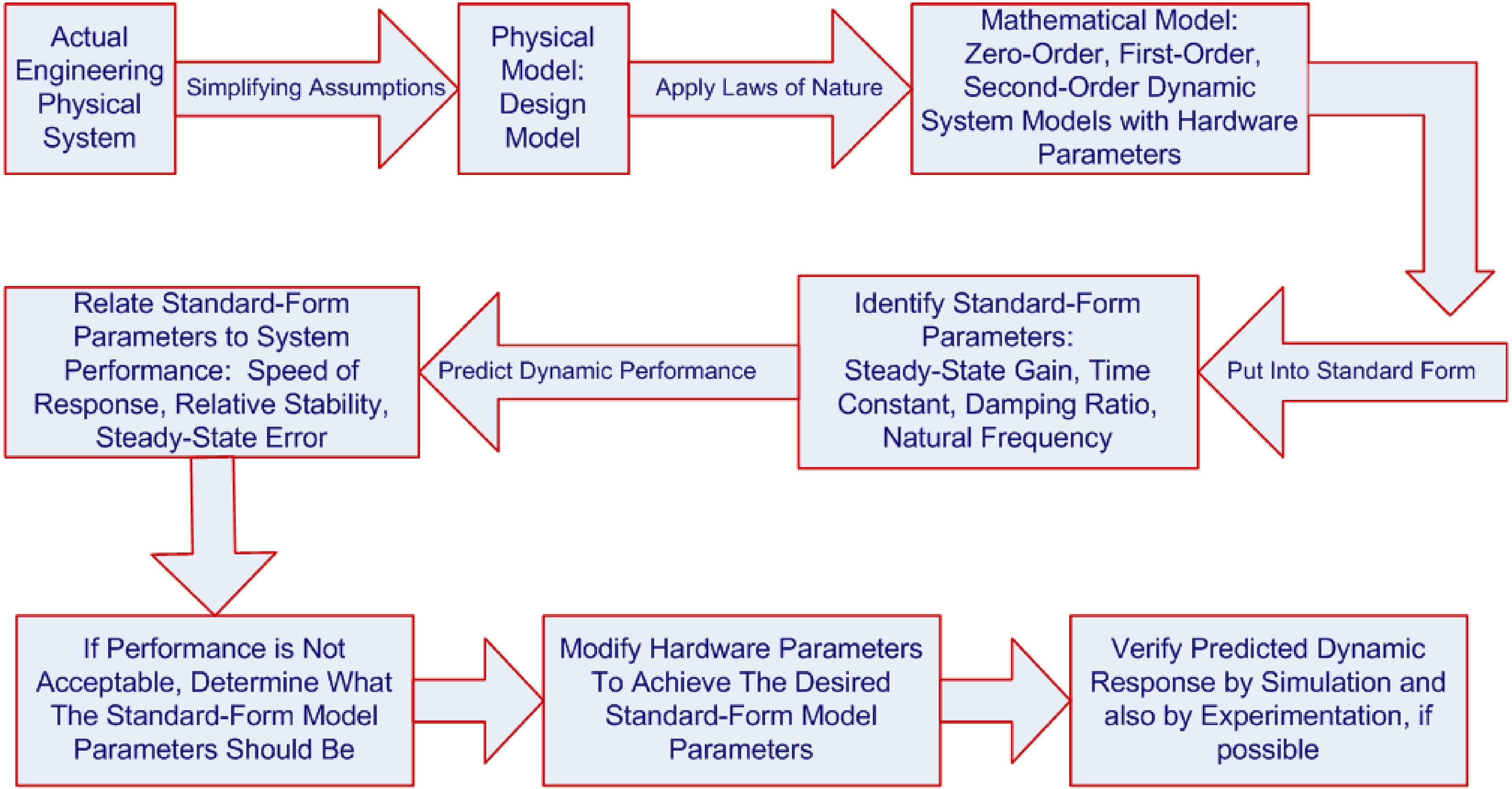


Diagram Showing How Physical Model Hardware Parameters Are Related to Physical Model Dynamic System Performance

- Experimental Determination of ζ and ω_n
  - ζ and ω_n can be obtained in a number of ways from step or frequency-response tests.
  - For an underdamped second-order system, the values of ζ and ω_n may be found from the relations:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \sqrt{\frac{1}{\left(\frac{\pi}{\log_e\left(M_p\right)}\right)^2 + 1}}$$

$$T = \frac{2\pi}{\omega_d} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi}{T\sqrt{1-\zeta^2}}$$

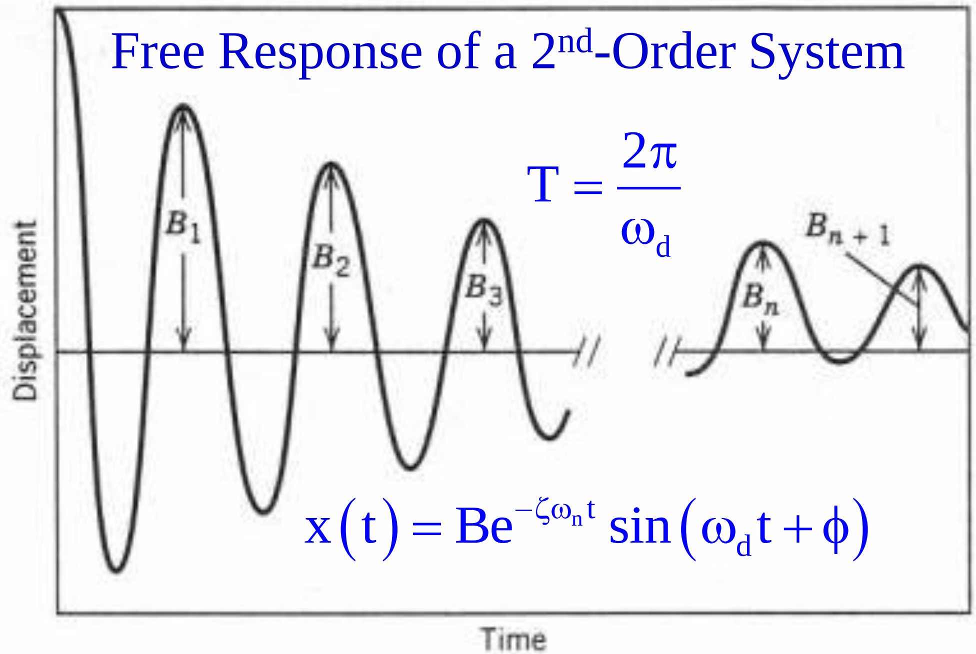
– Logarithmic Decrement δ is the natural logarithm of the ratio of two successive amplitudes.

$$\delta = \ln\left(\frac{x(t)}{x(t+T)}\right) = \ln\left(e^{\zeta\omega_n T}\right) = \zeta\omega_n T$$

$$= \frac{\zeta\omega_n 2\pi}{\omega_d} = \frac{\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$$

$$\delta = \frac{1}{n}\ln\frac{B_1}{B_{n+1}}$$



- If several cycles of oscillation appear in the record, it is more accurate to determine the period T as the average of as many distinct cycles as are available rather than from a single cycle.
- If a system is strictly linear and second-order, the value of n is immaterial; the same value of ζ will be found for any number of cycles. Thus if ζ is calculated for, say, $n = 1, 2, 4$, and 6 and different numerical values of ζ are obtained, we know that the system is not following the postulated mathematical model.

- For over-damped systems ($\zeta > 1.0$), no oscillations exist, and the determination of ζ and ω_n becomes more difficult. Usually it is easier to express the system response in terms of two time constants.

– For the over-damped step response:

$$q_o = Kq_{is}\left[\begin{array}{l} 1-\dfrac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} \\ \qquad\qquad +\dfrac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \end{array}\right] \qquad \zeta>1$$

$$\frac{q_o}{Kq_{is}} = \frac{\tau_1}{\tau_2-\tau_1}e^{-\frac{t}{\tau_1}} - \frac{\tau_2}{\tau_2-\tau_1}e^{-\frac{t}{\tau_2}} + 1$$

– where

$$\tau_1 \triangleq \frac{1}{\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n} \qquad \tau_2 \triangleq \frac{1}{\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n}$$

– To find τ_1 and τ_2 from a step-function response curve, we may proceed as follows:

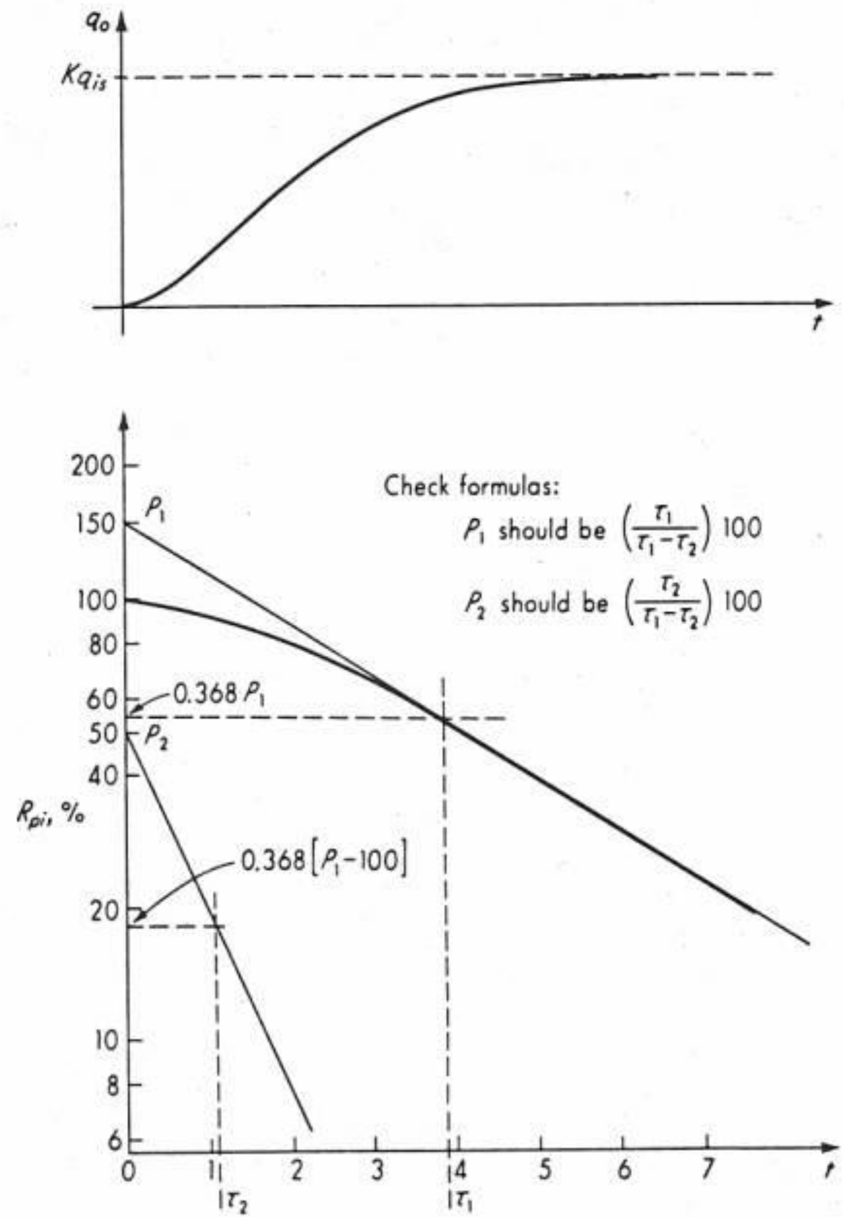
- Define the percent incomplete response R_{pi} as:

$$R_{pi} \triangleq \left(1 - \frac{q_o}{Kq_{is}}\right)100$$

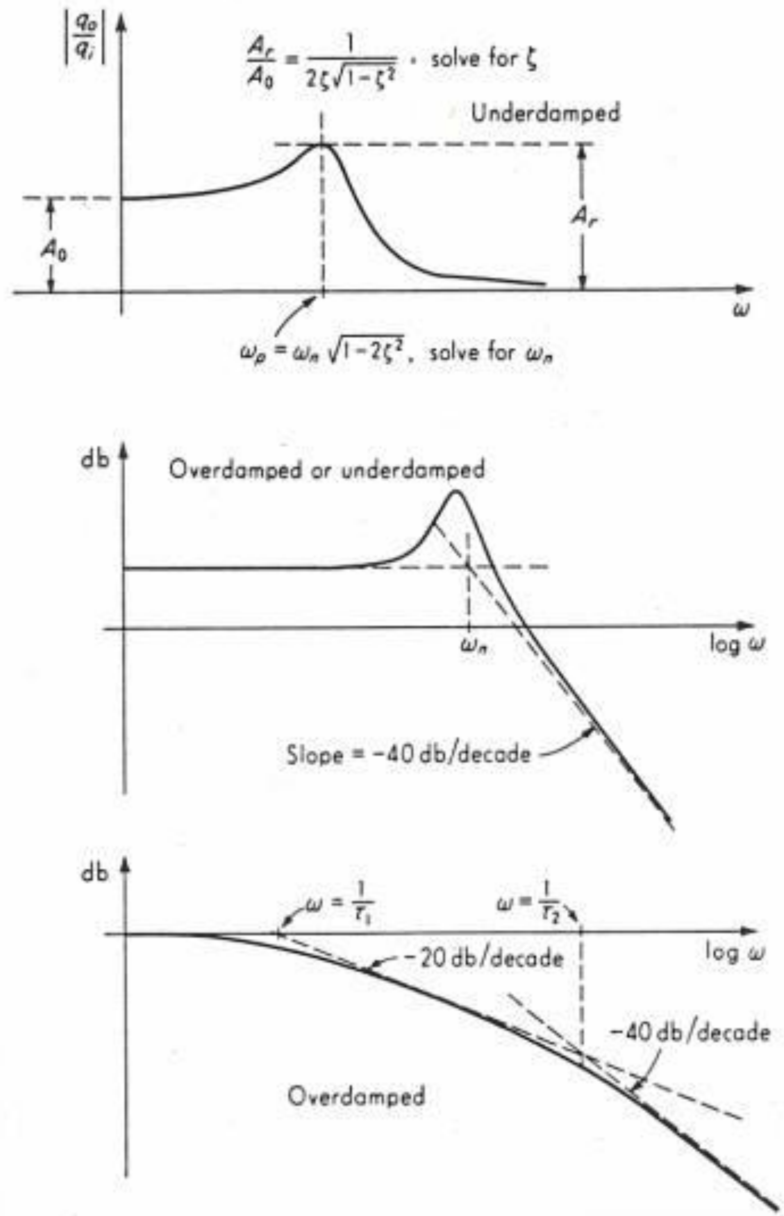
- Plot R_{pi} on a logarithmic scale versus time *t* on a linear scale. This curve will approach a straight line for large *t* if the system is second-order. Extend this line back to $t = 0$, and note the value P_1 where this line intersects the R_{pi} scale. Now, τ_1 is the time at which the straight-line asymptote has the value $0.368P_1$.

- Now plot on the same graph a new curve which is the difference between the straight-line asymptote and R_{pi}. If this new curve is not a straight line, the system is not second-order. If it is a straight line, the time at which this line has the value $0.368(P_1-100)$ is numerically equal to τ_2.
- Frequency-response methods may also be used to find τ_1 and τ_2.

Step-Response Test for Overdamped Second-Order Systems



Frequency-Response Test of Second-Order Systems



- Dynamic System Exercise
  - An underdamped 2^{nd}-order system model has the following transfer function:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\varsigma\omega_n \pm i\omega_n\sqrt{1-\varsigma^2}$$

$$s_{1,2} = -\sigma \pm i\omega_d$$

  - Part 1:
    - Using the properties and formulas for 2^{nd}-order systems, discuss the relationships between the step-response-parameters rise time, settling time, and overshoot, and the frequency-response-parameters bandwidth and peak amplitude as the model parameters vary. Use plots as needed in your presentation.

- Suggestion: Pick a base system. Generate 4 families of plots
  - ω_d constant, vary σ
  - σ constant, vary ω_d
  - ω_n constant, vary ζ
  - ζ constant, vary ω_n
- Show both time-response and frequency-response plots. Include discussion.

– Part 2:

- Investigate the effects on the time (step) response and frequency response of adding a real pole or a real zero to the 2nd-order transfer function. The pole and zero are added separately. In classical deign using root-locus or frequency-response techniques, real poles and zeros are added (lead, lag, lead-lag controllers) to modify system dynamics, and so it is important to have a good understanding of these effects. Use plots as needed in your presentation.

- Suggestion: Pick a base second-order system.
  - Add a negative real pole (s + p) to the transfer function and move the pole from the left towards the origin and describe its effect on the time-response and frequency-response plots.
  - Add a negative real zero (s + z) to the transfer function and move the zero from the left towards the origin and describe its effect on the time-response and frequency-response plots.

– Part 3:

- Now add a positive real zero to your base second-order system and evaluate the step response for the system. Explain your observations.
- Physically, what might cause a transfer function to have a right-half plane zero?

- Problem Solution

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + (\omega_d^2 + \sigma^2)}$$

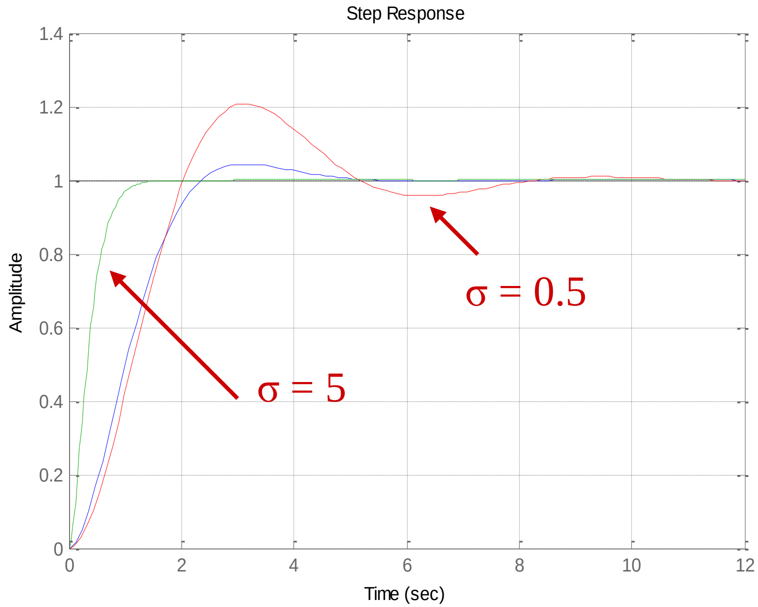
  - Base System

$$G(s) = \frac{2}{s^2 + 2s + 2}$$

$$\sigma = 1 \qquad \omega_n = \sqrt{2}$$

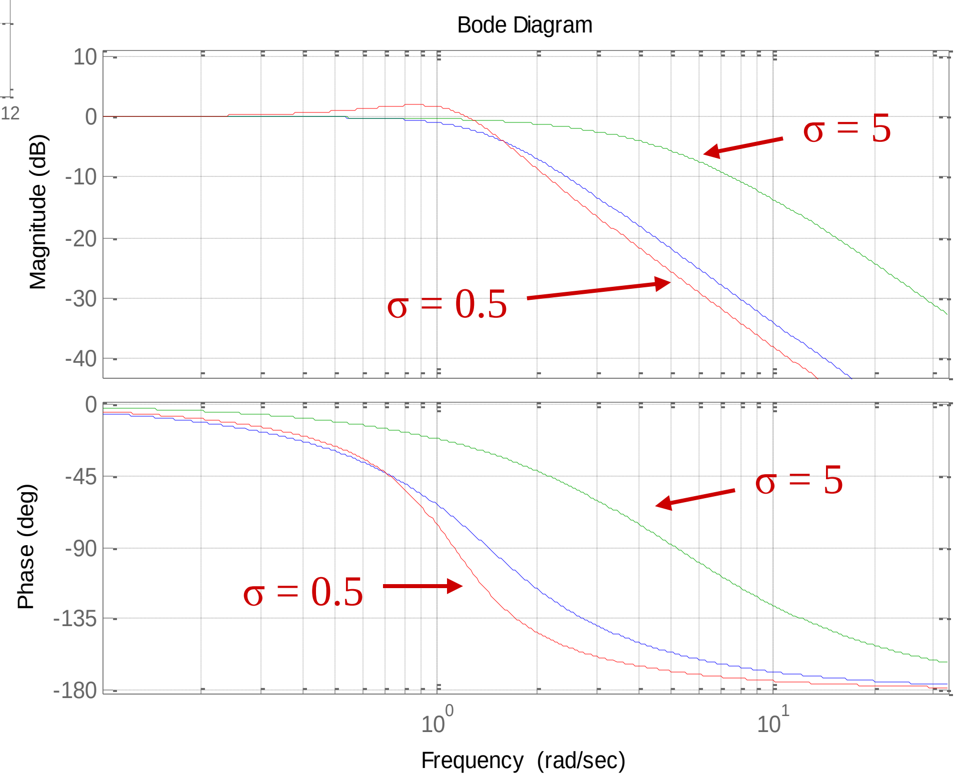
$$\omega_d = 1 \qquad \zeta = 0.707$$

  - Effects of σ:
    - $\omega_d = 1$, $\sigma = [0.5, 1, 5]$
  - Effects of ω_d:
    - $\sigma = 1$, $\omega_d = [0.5, 1, 5]$
  - Effects of ω_n:
    - $\zeta = 0.707$, $\omega_n = [0.5\sqrt{2}, \sqrt{2}, 5\sqrt{2}]$
  - Effects of ζ:
    - $\omega_n = \sqrt{2}$, $\zeta = [0.866, 0.707, 0.5]$



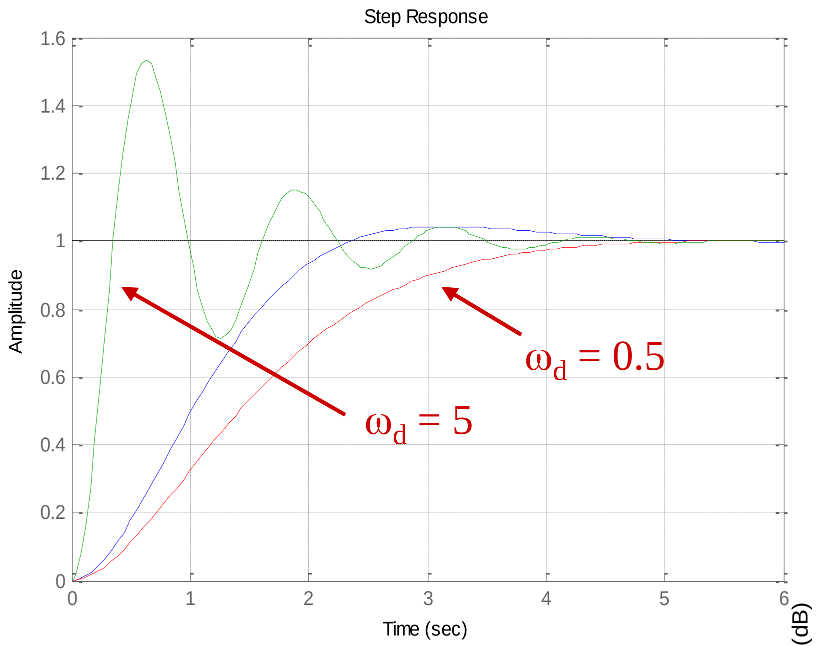
Effects of varying σ

$\omega_d = 1, \sigma = [0.5, 1, 5]$



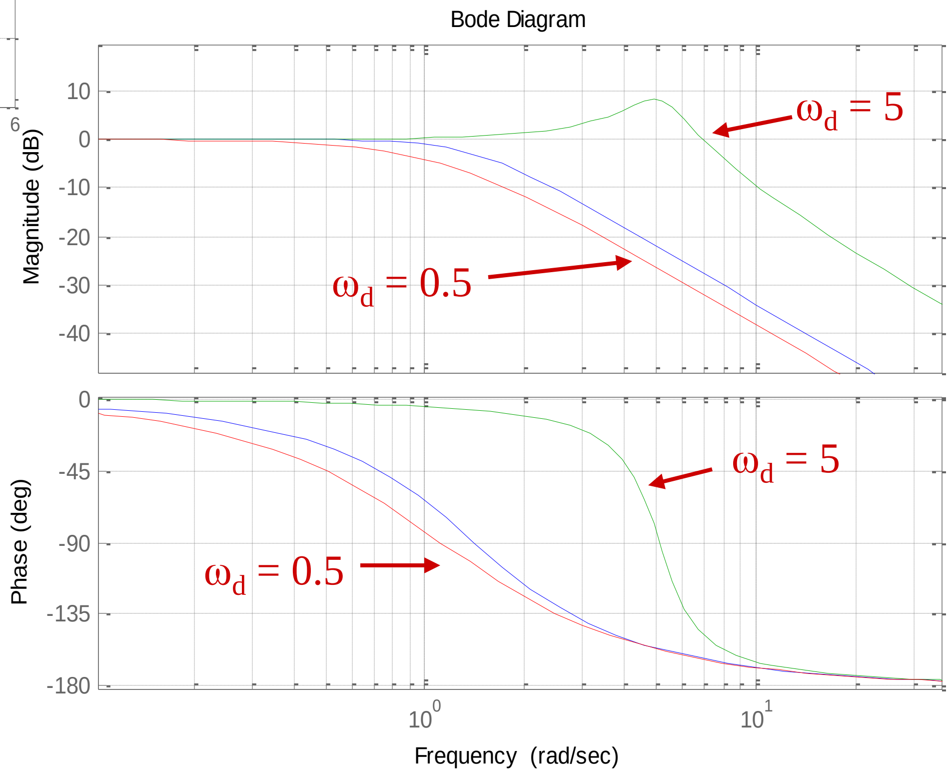
As σ increases:

- t_s decreases
- t_r decreases
- M_p decreases
- BW increases



Effects of varying ω_d

$\sigma = 1, \omega_d = [0.5, 1, 5]$

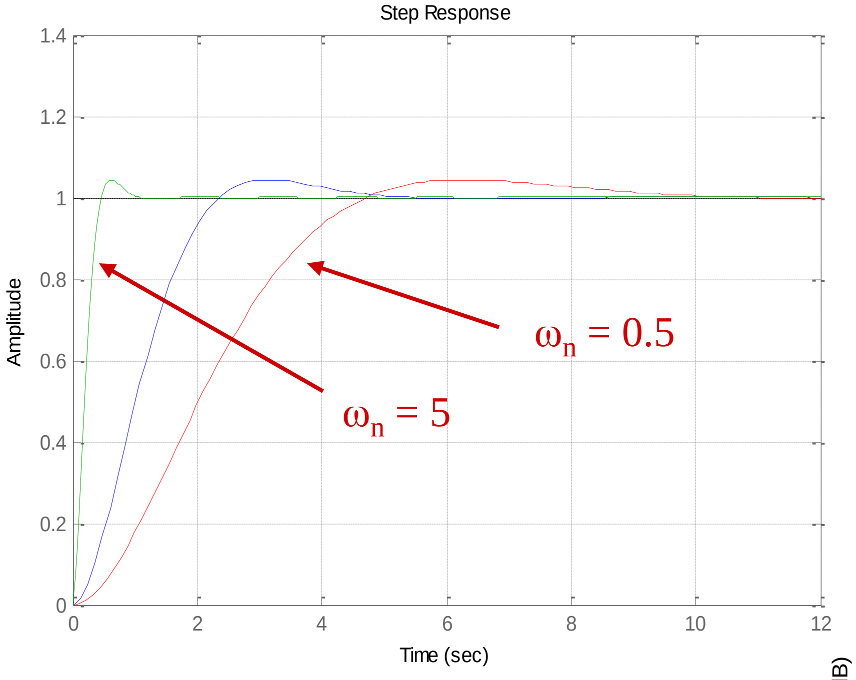


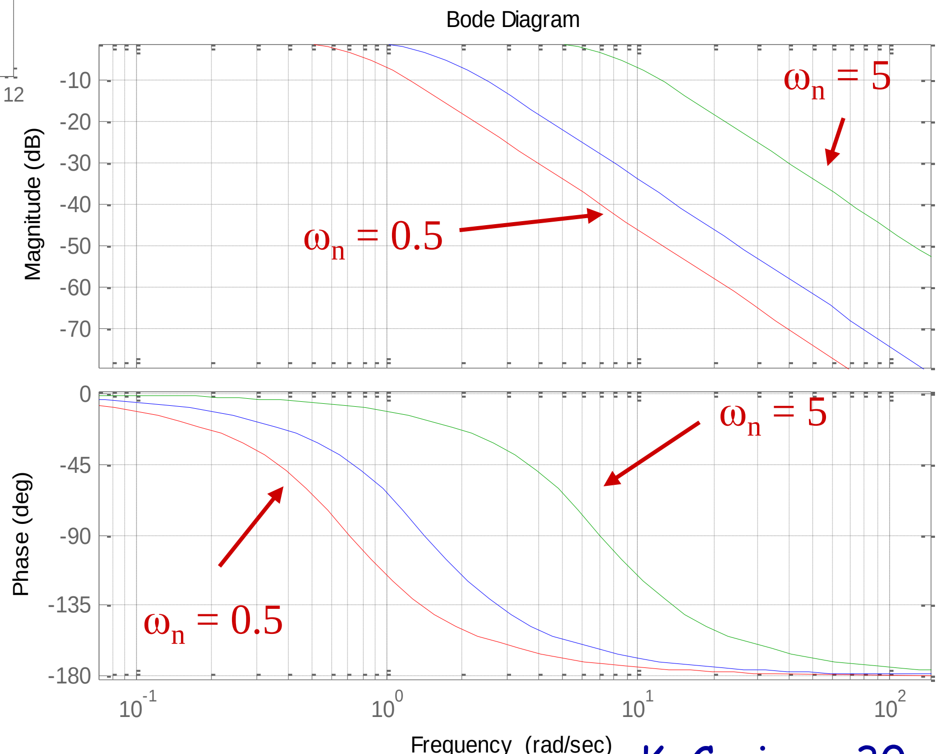
As ω_d increases:

- t_s is fixed
- t_r decreases
- M_p increases
- BW increases

Effects of varying ω_n

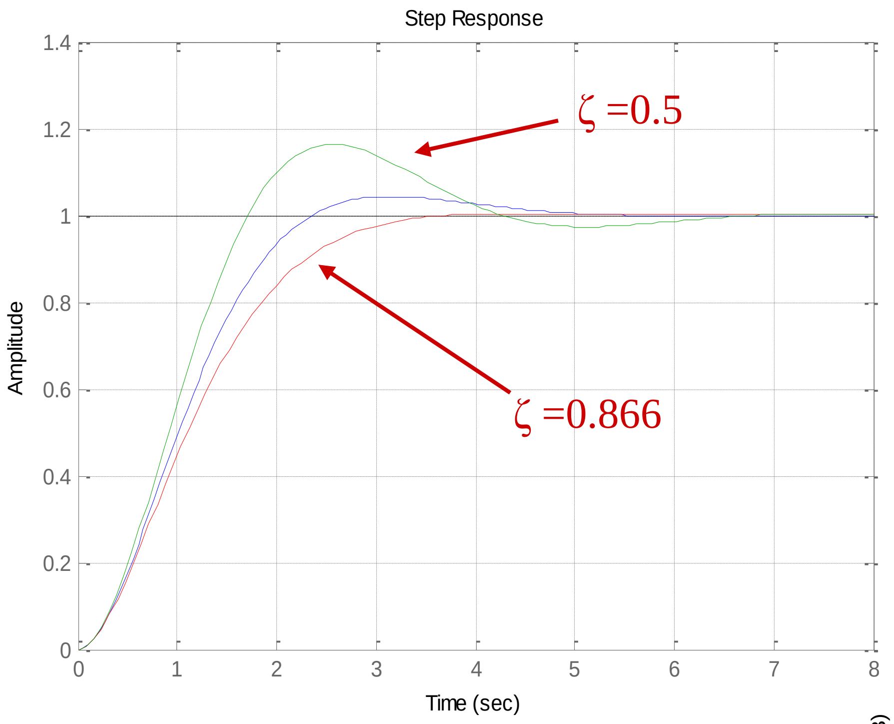
$\zeta = 0.707, \ \omega_n = [0.5\sqrt{2}, \sqrt{2}, 5\sqrt{2}]$





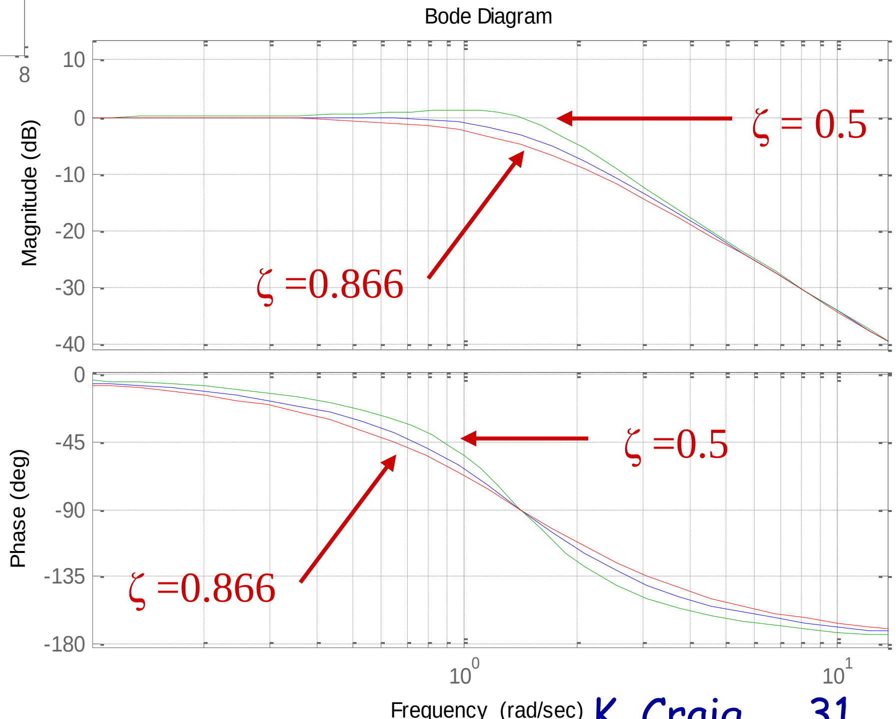
As ω_n increases:

- t_s decreases
- t_r decreases
- M_p is fixed
- BW increases



Effects of varying ζ

$\omega_n = \sqrt{2}, \zeta = [0.866, 0.707, 0.5]$



As ζ increases:

- t_s increases
- t_r decreases
- M_p increases
- BW increases

- Effect of an Additional LHP Pole

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + (\omega_d^2 + \sigma^2)}$$

– Base System

$$G(s) = \frac{2}{s^2 + 2s + 2}$$

$\sigma = 1$ $\omega_n = \sqrt{2}$

$\omega_d = 1$ $\zeta = 0.707$

– Additional Pole

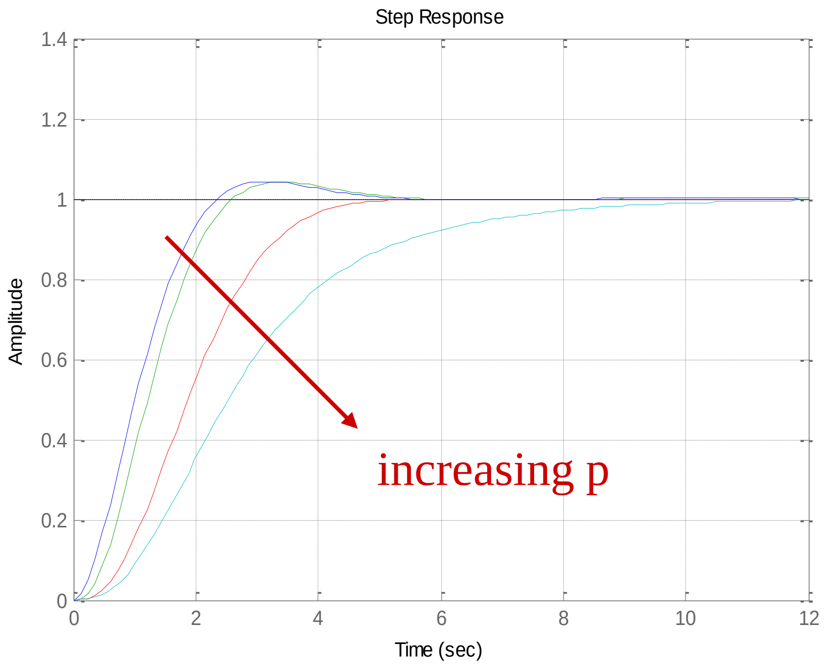
$$G(s) = \frac{2}{(ps+1)(s^2 + 2s + 2)}$$

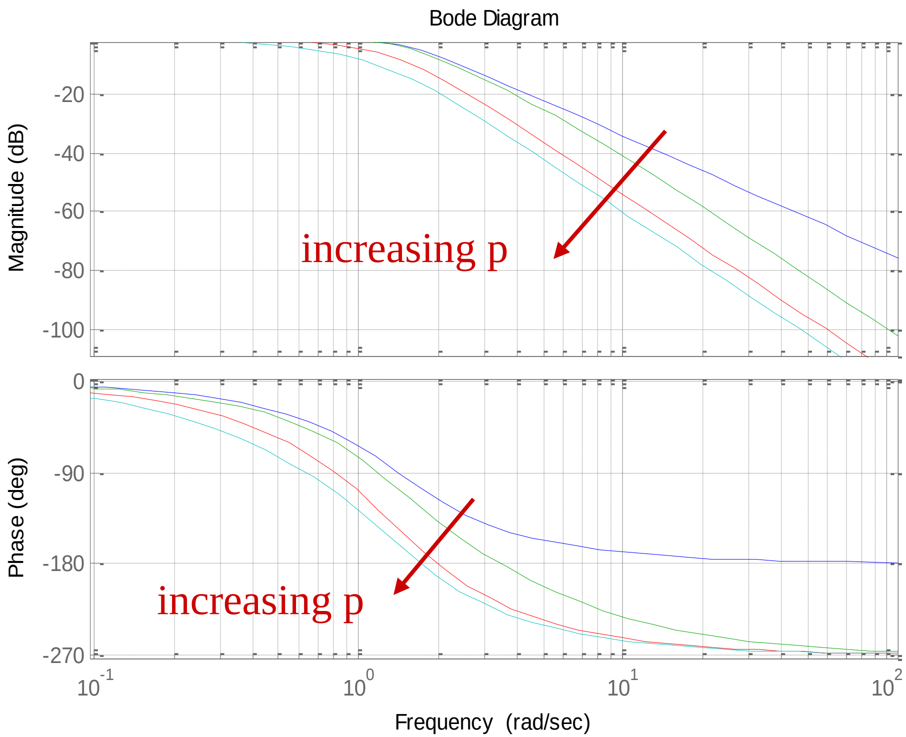
$$= \frac{1}{ps^3 + (2p+1)s^2 + (2p+2)s + 2}$$

$p = [0, 0.2,\ 1,\ 2]$

Effect of an Additional Pole

$$p = [0, 0.2,\ 1,\ 2]$$





As p increases (pole gets closer to the origin):

- t_s increases
- t_r increases
- M_p decreases to zero
- BW decreases

- Effect of a LHP Zero

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_d^2+\sigma^2}{s^2+2\sigma s+(\omega_d^2+\sigma^2)}$$

  - Base System

$$G(s)=\frac{2}{s^2+2s+2}$$

$$\sigma=1 \qquad \omega_n=\sqrt{2}$$

$$\omega_d=1 \qquad \zeta=0.707$$

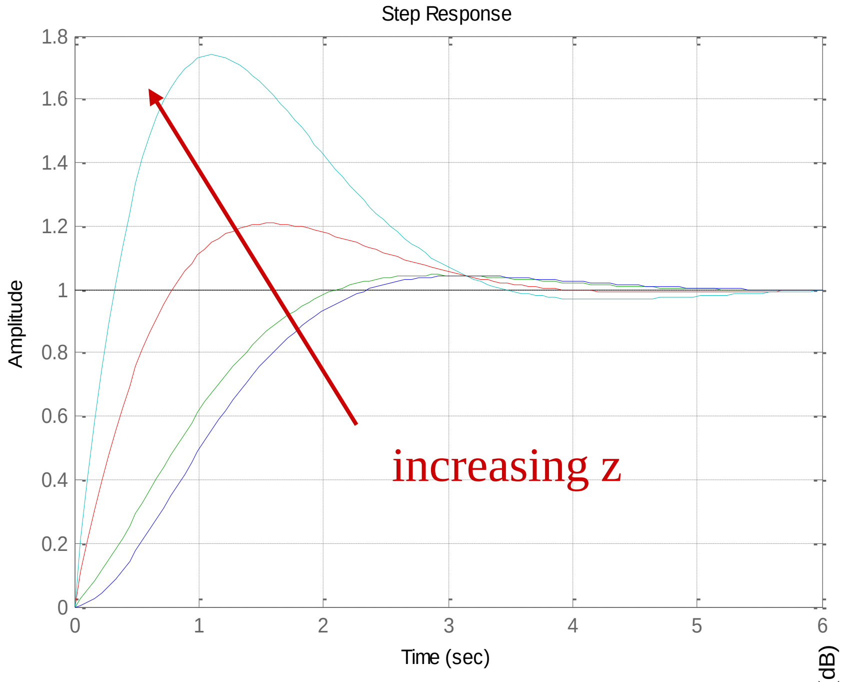
  - Add a Zero

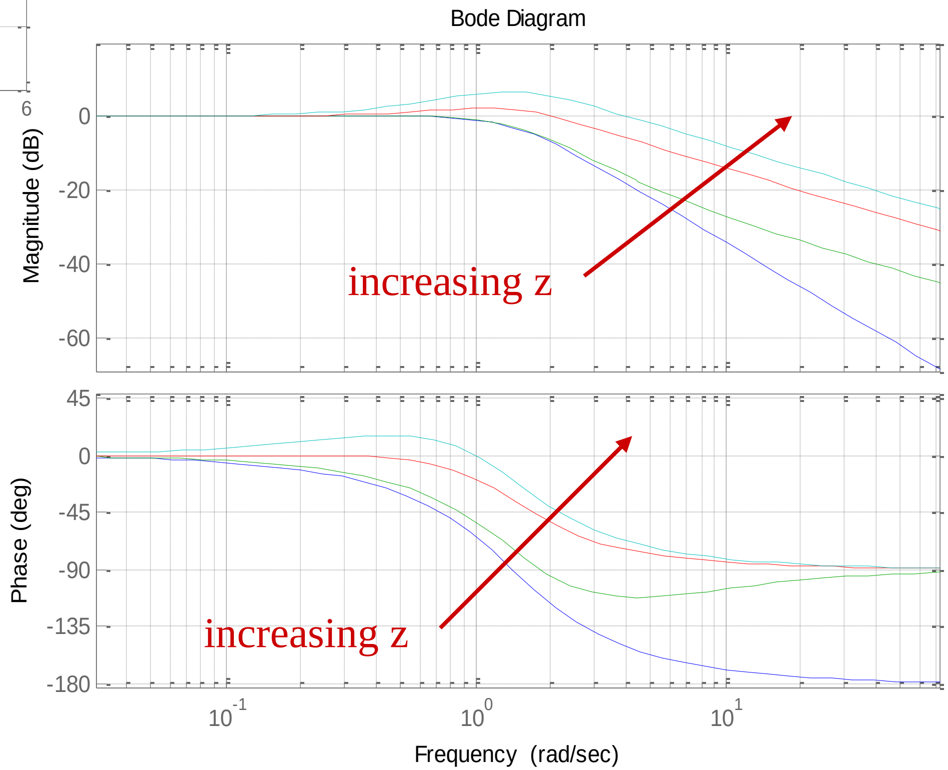
$$G(s)=\frac{2(zs+1)}{(s^2+2s+2)}$$

$$z=[0, 0.2,\ 1,\ 2]$$

Effect of a LHP Zero

$$z = [0, 0.2,\ 1,\ 2]$$





As z increases (zero gets closer to the origin):

- t_s increases
- t_r decreases
- M_p increases
- BW increases

- <u>Effect of a RHP Zero</u>

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_d^2+\sigma^2}{s^2+2\sigma s+(\omega_d^2+\sigma^2)}$$

– Base System

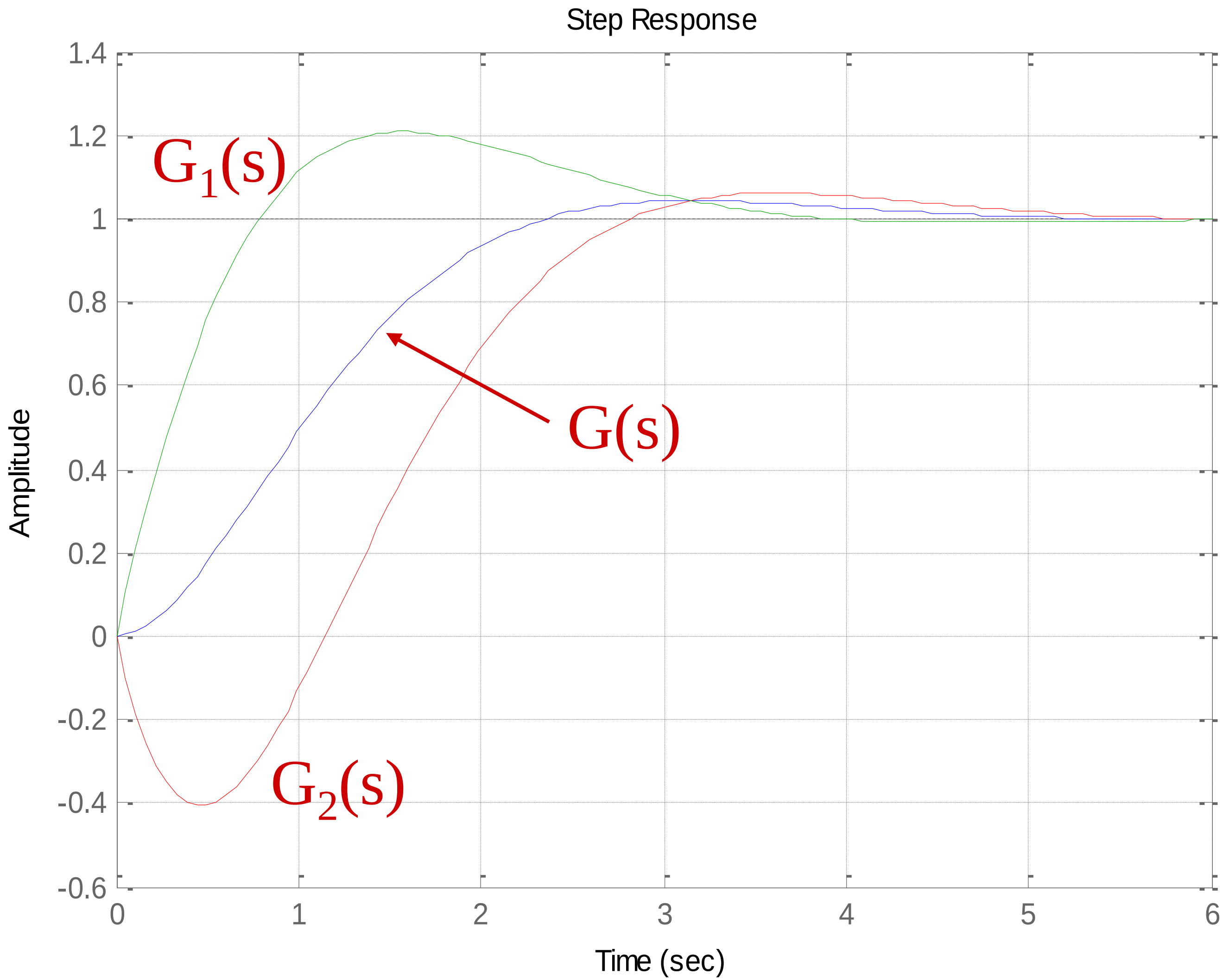
$$G(s)=\frac{2}{s^2+2s+2}\qquad \sigma=1\qquad \omega_n=\sqrt{2}\qquad \omega_d=1\qquad \zeta=0.707$$

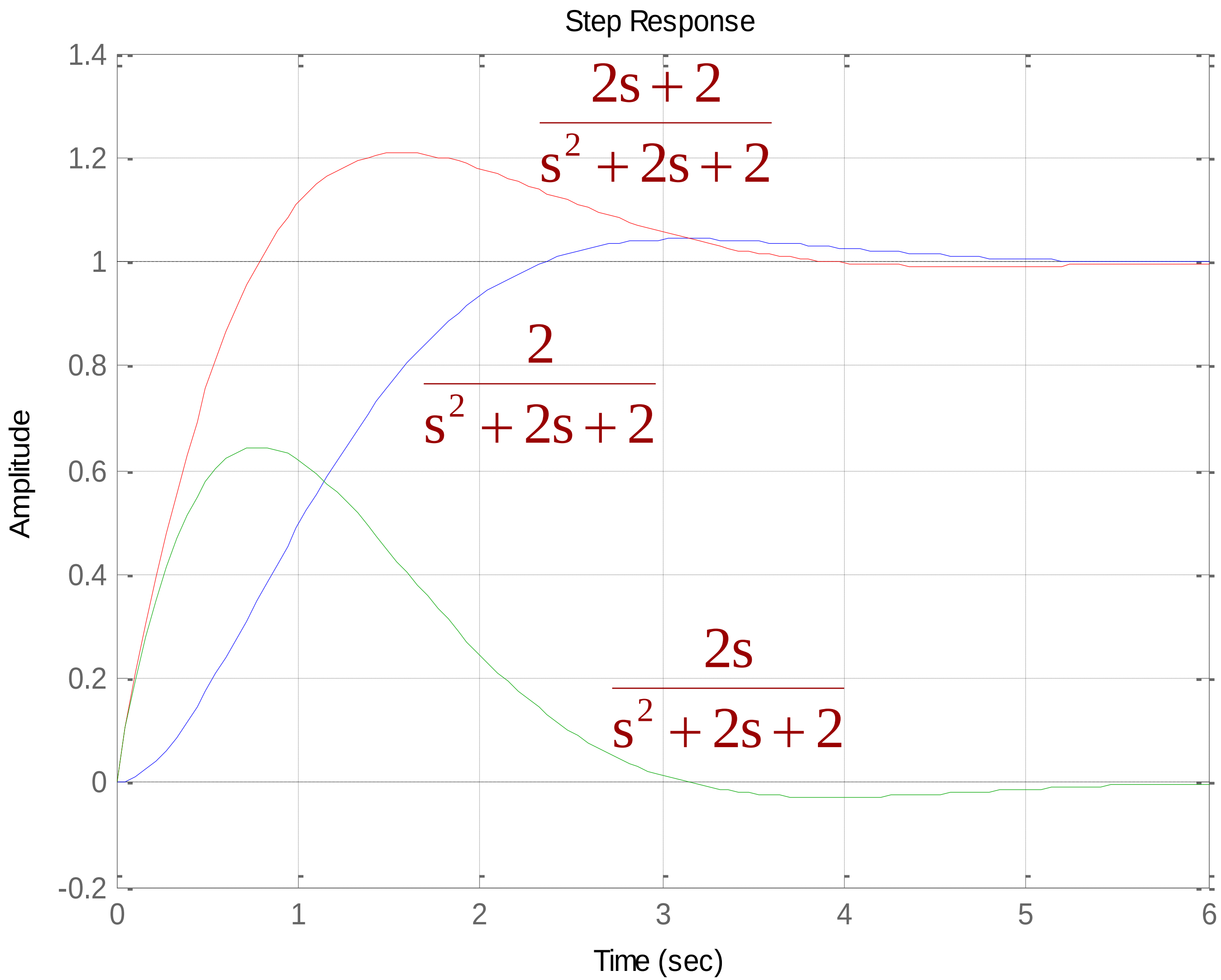
– Add a RHP Zero

$$G(s)=\frac{2}{(s^2+2s+2)}$$

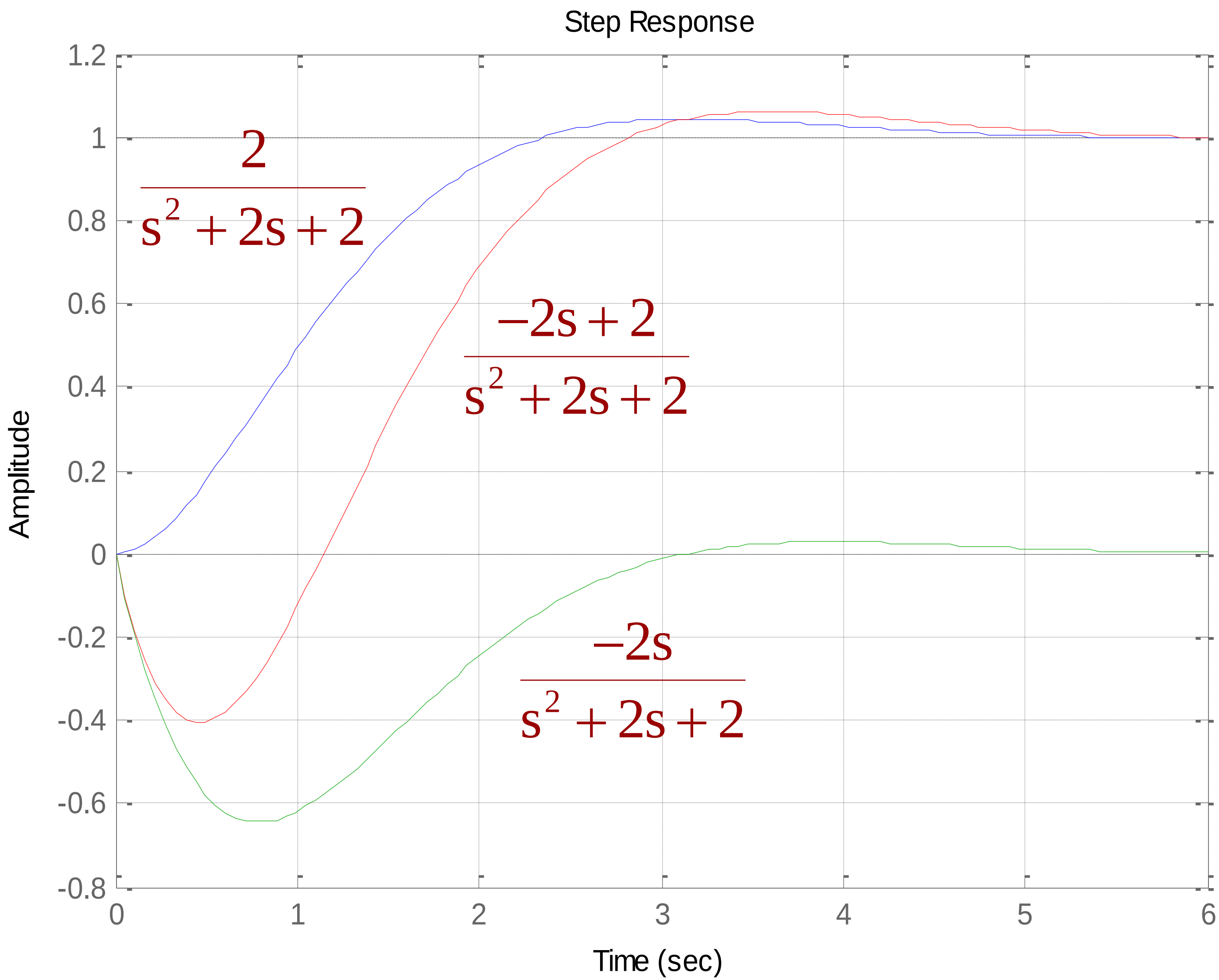
$$G_1(s)=\frac{2}{(s^2+2s+2)}+\frac{2s}{(s^2+2s+2)}=\frac{(2s+2)}{(s^2+2s+2)}$$ G(s) plus its derivative

$$G_2(s)=\frac{2}{(s^2+2s+2)}-\frac{2s}{(s^2+2s+2)}=\frac{(-2s+2)}{(s^2+2s+2)}$$ G(s) minus its derivative


Step Response
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
1
2
3
4
5
6
$G_1(s)$
$G(s)$
$G_2(s)$



Step Response
$\frac{2s+2}{s^2+2s+2}$
$\frac{2}{s^2+2s+2}$
$\frac{2s}{s^2+2s+2}$
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
1
2
3
4
5
6


Step Response
Amplitude
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
1
2
3
4
5
6
$\frac{2}{s^2+2s+2}$
$\frac{-2s+2}{s^2+2s+2}$
$\frac{-2s}{s^2+2s+2}$