Plane Kinetics of a Rigid Body

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Plane Motion:
- contains mass center
- forces projected onto plane of motion
- symmetry with respect to plane of motion

I. Force, Mass, and Acceleration Method

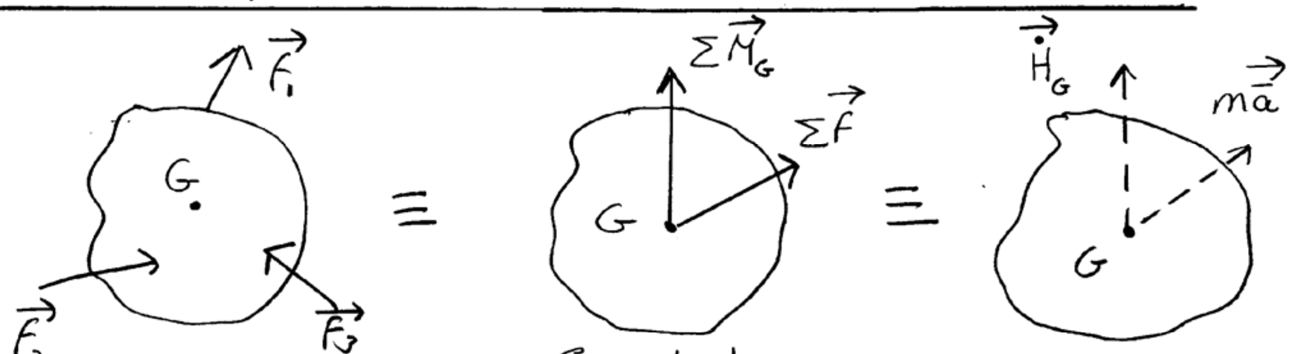
II. Work-Energy Method

III. Impulse-Momentum Method

3 Degrees of Freedom

Free Body Diagram – MOST IMPORTANT !!!

I. Force, Mass, and Acceleration Method



Free-Body Diagram — Equivalent Force-Couple System — Effect Diagram

$$\boxed{\begin{aligned}\Sigma\vec{F} &= m\vec{\bar{a}}\\ \Sigma\vec{M}_G &= \dot{\vec{H}}_G\end{aligned}}$$

$\sum \vec{F}$ = external forces acting on rigid body

$\sum \vec{M}_G$ = resultant moment of external forces about mass center G

$\dot{\vec{H}}_G = \frac{{}^{R}d}{dt}\vec{H}_G$ (R – ground reference frame)

$\vec{H}_G$ = angular momentum of rigid body about G

$\vec{\bar{a}}$ = absolute acceleration of mass center G

Plane Motion Equations

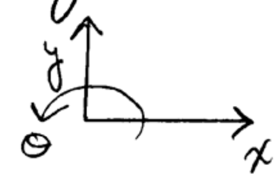
$$\bar{H} = \bar{I}\omega$$

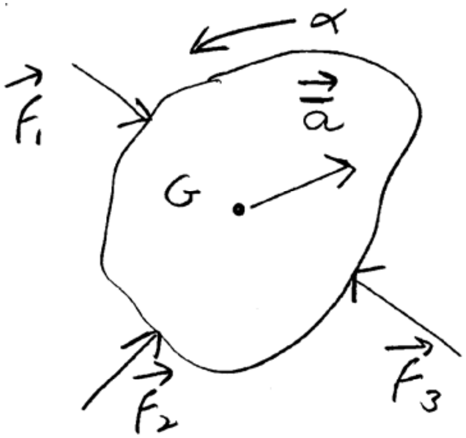
$$\bar{I} = \int \rho^2 \, dm$$

$\bar{I}$ = mass moment of inertia of body about z axis through G
= constant property of body and is a measure of the radial distribution of mass around z axis through G.
= measure of rotational inertia

$$\boxed{\begin{aligned} \sum f_x &= m\bar{a}_x \\ \sum f_y &= m\bar{a}_y \\ \sum \bar{M} &= \bar{I}\alpha \end{aligned}}$$

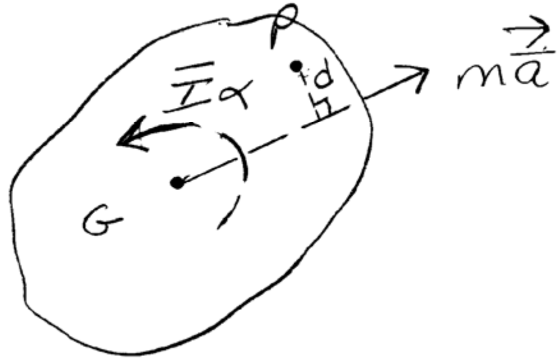
3 independent scalar equations





FBD

$\equiv$



Effect Diagram

$$\vec{a}\begin{cases} x-y \\ n-t \\ r-\theta \end{cases}$$

components

Plane Motion

For <u>any arbitrary point P</u>, moving or fixed, on or off the body, we may write:

$$\sum \vec{M}_P = \dot{\vec{H}}_G + \left(\vec{r}_{PG} \times m\vec{\bar a}\right)$$

For the plane-motion problem shown above, we may write:

$$\sum M_P = \bar I \alpha + m \bar a d$$

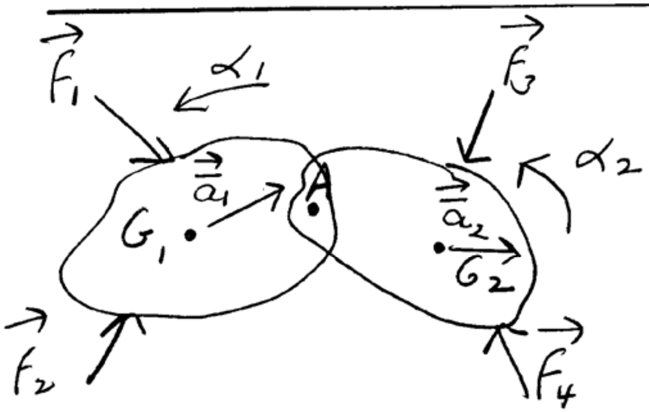
Principle of Moments

Alternative Equation

$$\left.\begin{aligned} \sum F_x &= m\bar a_x \\ \sum F_y &= m\bar a_y \\ \sum \bar M &= \bar I \alpha \end{aligned}\right\} \text{Basic independent scalar equations}$$

Interconnected Bodies

Analyze as an entire system or separate bodies and analyze individually.



FBD

$\equiv$

Effect Diagram

The Principle of Moments can be applied to the entire system.

$$\sum \vec{F} = m_1 \vec{\bar{a}}_1 + m_2 \vec{\bar{a}}_2$$

$$\sum M_P = \bar{I}_1\alpha_1 + \bar{I}_2\alpha_2 + m_1\bar{a}_1 d_1 + m_2 \bar{a}_2 d_2$$

d_1, d_2 = $\perp$ distances from point P to lines of action of $\vec{\bar{a}}_1$ and $\vec{\bar{a}}_2$, respectively. Observe correct direction ($\curvearrowright$ or $\curvearrowleft$).

In general: $\sum \vec{F} = \sum m\vec{\bar{a}}$ $\qquad \sum M_P = \sum \bar{I}\alpha + \sum m\bar{a}d$

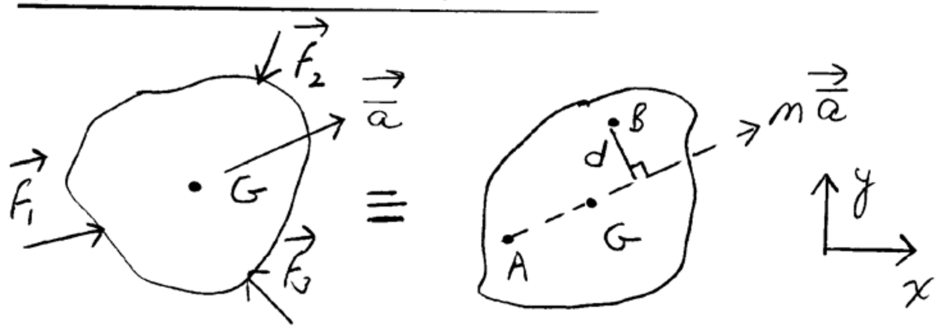
Special Cases:

(a) Translation (rectilinear and curvilinear)

$\alpha = 0$ (No angular motion)

$$\sum \vec{F} = m\vec{\bar{a}}$$
$$\sum \bar{M} = 0$$

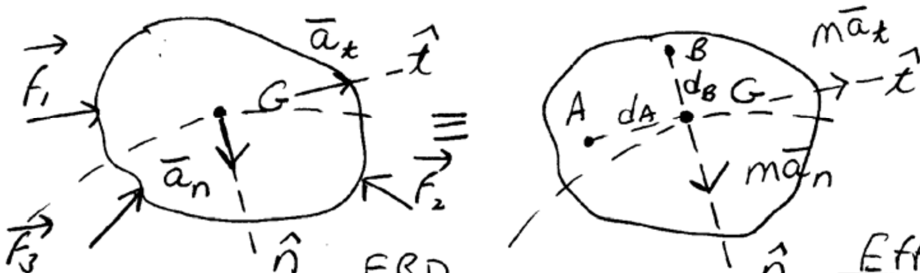
Rectilinear Translation



FBD Effect Diagram

$$\sum F_x = m\bar{a}_x$$
$$\sum F_y = m\bar{a}_y$$
$$\left.\begin{array}{l}\sum \bar{M} = 0\\ \sum M_A = 0\\ \sum M_B = m\bar{a}d\end{array}\right\}\text{alternatives}$$

Curvilinear Translation

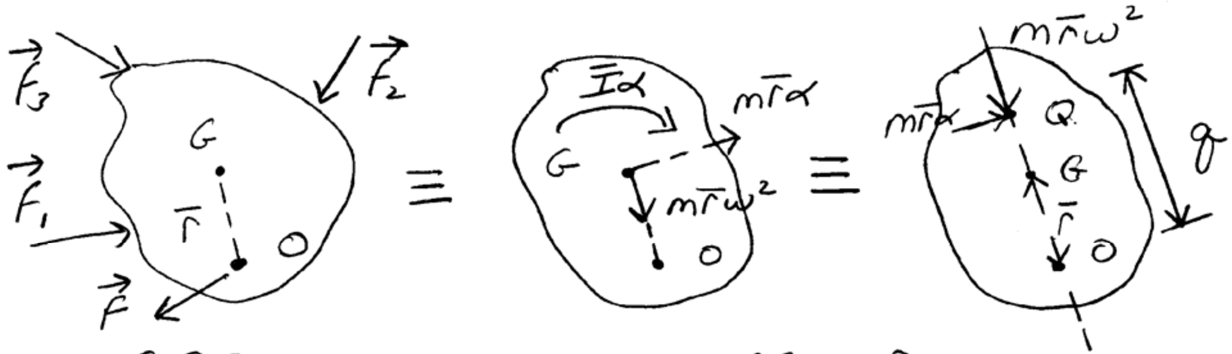


FBD Effect Diagram

$$\sum F_n = m\bar{a}_n$$
$$\sum F_t = m\bar{a}_t$$
$$\left.\begin{array}{l}\sum \bar{M} = 0\\ \sum M_A = m\bar{a}_n d_A\\ \sum M_B = m\bar{a}_t d_B\end{array}\right\}\text{alternatives}$$

(b) Fixed-Axis Rotation

Point O is location of fixed axis



Q = Center of Percussion (Unique point for a body with a fixed point O.)

$$\Sigma F_n = m\bar{r}\omega^2$$

$$\Sigma F_t = m\bar{r}\alpha$$

($\bar{r} = 0$ if O and G coincide)

$$\Sigma \bar{M} = \bar{I}\alpha$$

$$\Sigma M_o = \bar{I}\alpha + m\bar{r}^2\alpha$$

$$= (\bar{I} + m\bar{r}^2)\alpha = I_o\alpha$$

$$\Sigma M_Q = 0 \longrightarrow \left\{ \begin{aligned} (m\bar{r}\alpha)\bar{r} + \bar{I}\alpha &= (m\bar{r}\alpha)q \\ m\bar{r}^2\alpha + \bar{I}\alpha &= m\bar{r}\alpha q \\ \bar{r}^2 + \bar{k}^2 &= \bar{r}q \\ q &= \frac{\bar{r}^2 + \bar{k}^2}{\bar{r}} \\ q &= \frac{k_o^2}{\bar{r}} \end{aligned} \right.$$

Parallel Axis Theorem:

$$I_o = \bar{I} + m\bar{r}^2$$

$$m k_o^2 = m\bar{k}^2 + m\bar{r}^2$$

$$k_o^2 = \bar{k}^2 + \bar{r}^2$$

II. Work-Energy Method

(a) Work of a Force and Couple

$$U = \int \vec{F} \cdot d\vec{r} = \int F_t \, ds \qquad U = \int M \, d\theta$$

(b) Kinetic Energy

(i) Translation $T = \frac{1}{2} m v^2$

(ii) Fixed-Axis Rotation $T = \frac{1}{2} I_o \omega^2$

(iii) General Plane Motion $T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \bar{I} \omega^2$

(If point C = instantaneous center of zero velocity

$T = \frac{1}{2} I_c \omega^2$)

(c) Work-Energy Equation

$$U = \Delta T \quad \text{or} \quad U = \Delta T + \Delta V_g + \Delta V_e$$

This may be applied to a single rigid body or to an interconnected system of rigid bodies.

(d) Power = time rate at which work is performed $\boxed{8}$

$$P = \frac{dT}{dt} = \frac{d}{dt}\left[\frac{1}{2} m \vec{v}\cdot\vec{v} + \frac{1}{2}\bar{I}\omega^2\right]$$

$$= \frac{1}{2} m (\vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a}) + \bar{I}\omega\dot{\omega}$$

$$= m\vec{a}\cdot\vec{v} + \bar{I}\omega\dot{\omega}$$

$$= \Sigma\vec{F}\cdot\vec{v} + \Sigma\bar{M}\,\omega$$

III. Impulse - Momentum Method

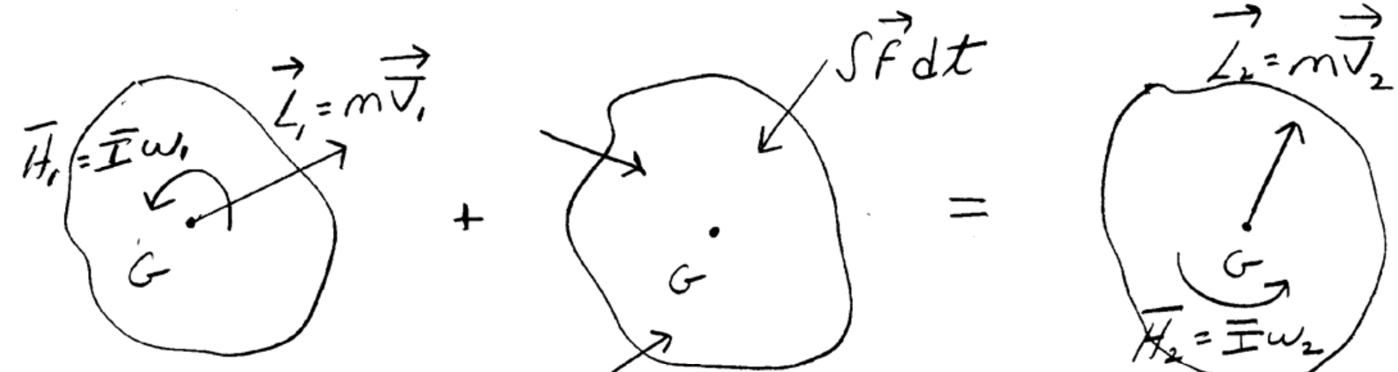
(a) Linear Momentum $\vec{L} = m\vec{V}$

(b) Angular Momentum $\vec{H} = \bar{I}\omega$

(c) $\Sigma \vec{F} = \frac{d}{dt}\vec{L}$ $\int_{t_1}^{t_2} \Sigma \vec{F}\, dt = \vec{L}_2 - \vec{L}_1$

(Linear Impulse)

(d) $\Sigma \bar{M} = \frac{d\bar{H}}{dt}$ $\int_{t_1}^{t_2} \Sigma \bar{M}\, dt = H_2 - H_1$



System Momenta time t_1 System External Impulses t_1 to t_2 System Momenta time t_2

3 Equations can be derived from these diagrams :

- Sum and equate x components of momenta and impulses
- Sum and equate y components of momenta and impulses
- Sum and equate moments of vectors about any point.

(e) Systems of Rigid Bodies

The impulse-momentum equations can be applied to each body separately or to the system as a whole.