Mathieu's Equation

Siva Srinivas Kolukula
Structural Mechanics Laboratory
IGCAR, Kalpakkam
INDIA
allwayzitzme@gmail.com

1. Introduction

Mathieu Equations are encountered in various areas of physics and engineering. Certain problems in theoretical physics lead to Mathieu equation, particularly the problem of the propagation of electromagnetic waves in a medium with a periodic structure, the problem of motion of electrons in a crystal lattice in the quantum theory of metals. This equation is also encountered in the investigation of stability in the nonlinear oscillations, namely inverted pendulum, parametric oscillator etc. The Mathieu equation occurs in the study of parametric excitation and parametric resonance in the theory of nonlinear resonances.

2. Mathieu Equation

The Mathieu equation is a second-order homogenous differential equation, given as

$$\ddot{x} + (\delta + 2\epsilon \cos 2t)x = 0 \tag{1}$$

Where δ and ε are regarded as parameters reflecting the system properties. The Mathieu's equation is a special case of Hill's equation. The solutions of the Mathieu's equation are complex and called as Mathieu's functions. Here, we are not interested in deriving these solutions but interested in knowing how these solutions behave as the system parameters δ and ε vary.

3. Floquet's Theory

According to the Floquet's theory equation (1) has two independent solutions of the form $e^{\gamma_1 t}\varphi(t)$ and $e^{\gamma_2 t}\varphi(t)$, where $\varphi(t)$ is periodic and γ_1 and γ_2 are functions of the system parameters δ and ε called as characteristic exponents. When $\gamma_1 = \gamma_2 = \gamma$, one solution is of the form $e^{\gamma t}\varphi(t)$ while the second is of the form $te^{\gamma t}\varphi(t)$. One can consider a space defined by the system parameters δ and ε which consists of regions in which the real parts of γ_1 and γ_2 are negative, indicating bounded (stable) solutions, and regions in which the real part of γ_1 and γ_2 is positive indicating an unbounded (unstable) solution. On the boundaries between stable and unstable motions, the real parts of γ_1 and γ_2 are zero and $\gamma_1 = \gamma_2$. If $\gamma_1 = \gamma_2 = \gamma = 1$, the solution is periodic with period T = π and if $\gamma_1 = \gamma_2 = \gamma = -1$the solution is periodic with period 2T = 2π. Thus the boundaries are characterized by two independent solutions of the form $\varphi(t)$ and $t\varphi(t)$. Of these solutions, the periodic one is of great interest since, in determining such a solution, one can also determine the stability boundaries. The fact that the second independent solution is unstable indicates that the transition boundaries themselves must be included in the instability regions.

4. Solution by Harmonic Balance Method

To establish the regions of bounded and unbounded motions in the parameter plane we need only obtain the boundary curves. To determine the boundaries, the solution of equation (1) is considered as a Fourier series,

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (2)$$

Where a_n and b_n are the Fourier coefficients. The boundary curves in the parameter plane δ vs. ε, can be determined by introducing equation (2) into equation (1) and using the principle of harmonic balance.

According to the principle of harmonic balance, the coefficient of every cosine and sine term is to be equated to zero. This will yield and infinite set of homogenous, linear, algebraic equations in a's and b's. Inspection of these equations reveals that only evenly or oddly subscripted a's and b's are involved. Thus one obtains four sets of independent equations. Two sets of even equations for cosine and sine, and two sets of odd equations for cosine and sine. These four sets of equations are referred by even cosine, even sine, odd cosine and odd sine. The four sets of algebraic equations are:

Odd cosine: $(\delta - 1 + \varepsilon)a_1 + \varepsilon a_2 = 0$

$$\varepsilon(a_n + a_{n+2}) + [\delta - (2n+1)^2]a_{n+1} = 0, \qquad n = 1,2, \dots \dots \qquad (3)$$

Odd sine: $(\delta - 1 - \varepsilon)b_1 + \varepsilon b_2 = 0$

$$\varepsilon(b_n + b_{n+2}) + [\delta - (2n+1)^2]b_{n+1} = 0, \qquad n = 1,2, \dots \dots \qquad (4)$$

Even cosine: $\delta a_0 + \varepsilon a_1 = 0$

$$2\varepsilon a_0 + (\delta - 4)a_1 + \varepsilon a_2 = 0$$

$$[\delta - 4(n+1)^2]a_{n+1} + \epsilon(a_n + a_{n+2}) = 0, \qquad n = 1,2, \dots \qquad (5)$$

Even sine: $(\delta - 4)b_1 + \varepsilon b_2 = 0$

$$[\delta - 4(n+1)^2]b_{n+1} + \varepsilon(b_n + b_{n+2}) = 0, \qquad n = 1,2, \dots \dots \qquad (6)$$

5. Hill's Determinants

The set of above equations have non-trivial solution if the determinant of the matrix obtained by the coefficients a's and b's is zero. The determinants are of infinite order and are called Hill's infinite determinants. The Hill's infinite determinants are given as follows:

Odd cosine:

$$\begin{vmatrix} \delta - 1 + \varepsilon & \varepsilon & 0 & 0 & \dots \\ \varepsilon & \delta - 3^2 & \varepsilon & 0 & \dots \\ 0 & \varepsilon & \delta - 5^2 & \varepsilon & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = 0 \qquad (7)$$

Odd sine:

$$\begin{vmatrix} \delta - 1 - \varepsilon & \varepsilon & 0 & 0 & \dots \\ \varepsilon & \delta - 3^2 & \varepsilon & 0 & \dots \\ 0 & \varepsilon & \delta - 5^2 & \varepsilon & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = 0 \qquad (8)$$

Even Cosine:
$$\begin{vmatrix} \delta & \varepsilon & 0 & 0 & 0 & \dots \\ 2\varepsilon & \delta-2^2 & \varepsilon & 0 & 0 & \dots \\ 0 & \varepsilon & \delta-4^2 & 0 & 0 & \dots \\ 0 & 0 & \varepsilon & \delta-6^2 & \varepsilon & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0 \quad (9)$$

Even sine:
$$\begin{vmatrix} \delta-2^2 & \varepsilon & 0 & 0 & \dots \\ \varepsilon & \delta-4^2 & \varepsilon & 0 & \dots \\ 0 & \varepsilon & \delta-6^2 & \varepsilon & \dots \\ \dots & \dots & \dots & \dots & \end{vmatrix} = 0 \quad (10)$$

The Hill determinants are of infinite order; they are truncated to finite dimension before doing any analysis. The Hill determinants are tridiagonal matrix determinants. It is clear from the Hill determinants that δ always appears on the diagonal of the matrices and its value must render the determinant is equal to zero, one can invoke the analogy with the eigenvalue problem and refer δ as an eigenvalue. Then, for given values of ε, it is possible to calculate values of δ corresponding to periodic solutions by solving a family of eigenvalue problems. These eigenvalues are nothing but the characteristic exponents γ_1 and γ_2 discussed in section 3. For $\varepsilon \neq 0$, each determinant is unique; hence the four determinants give rise to four different families of curves in the parameter plane. Due to this uniqueness of eigenvalues, if any one determinant is zero, the remaining three are non zero and the solution of equation (1) connotes a sum of even or odd, sines or cosines according to which determinant is zero.

6. Code for stability chart of Mathieu's Equation

A code is written in MATLAB to solve the zeros of Hill determinants i.e. to find the eigenvalues of the matrices. For a given ε, eigenvalue analysis is done and the respective δ is calculated. The obtained values are plotted in the parameter plane δ vs. ε which gives the stability chart of the Mathieu equation. The stability chart can also be referred as Strutt diagram or Ince-strutt diagram. The eigenvalues are calculated using the inbuilt function eig of the MATLAB.

Figure 1 shows the stability chart for the Mathieu's equation (1). The intersection of the boundary curves with the δ axis can be obtained by simply setting ε=0 in the Hill determinants, which yields $\delta = n^2$. This can be easily verified from the stability chart shown in figure 2. The stability chart contains the regions of stability and instability. The boundary curves of period π and 2π lie alternatively in the stability chart. The region in the plane enclosed by the boundary curves with the same period π or 2π is an instable region and its solution is characterized by unbounded motion. The region enclosed by the boundary curves with different period i.e. π and 2π is a stable region and its solution is characterized bounded motion. It is clear from the figure that the regions are symmetric with respective to the δ axis.

For a given δ and ε, if the point lie's in the stable region then the system is stable and its solution is bounded w.r.t time. If the point lie's in the instable region, the system is unstable and its solution is unbounded, the solution keeps on increasing exponentially w.r.t time. This instability is called parametric instability or dynamic instability or parametric resonance. The system is said to be excited

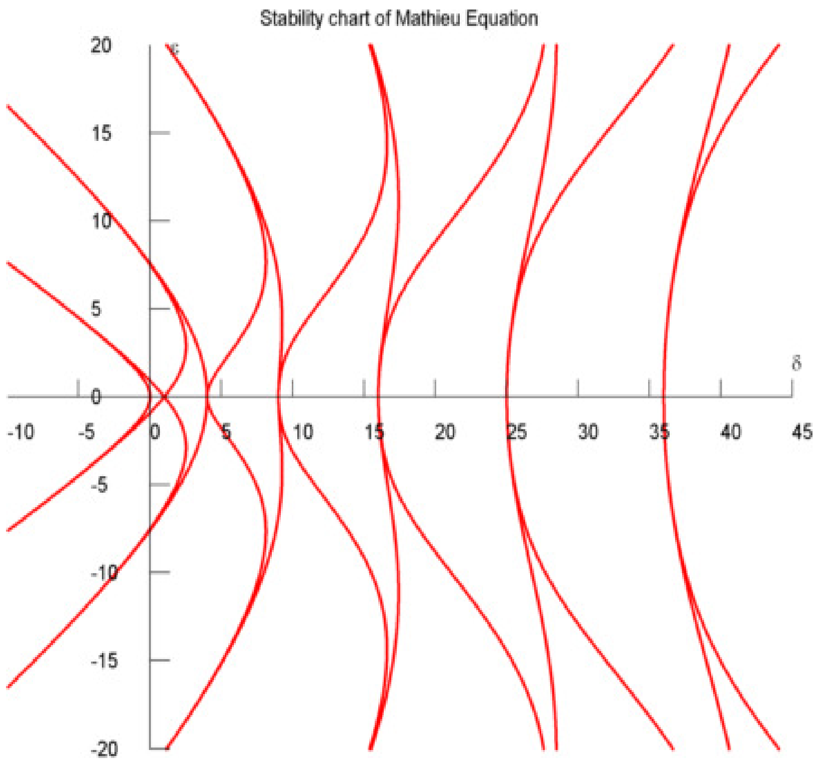

Figure 1: Stability Chart of Mathieu's Equation

parametrically and the periodic term in equation (1) is called parametric excitation. To know the behaviour of system under parametric excitation and use of stability charts refer [1].

For any discussions, advice, bugs, developing the code please feel free to mail me. Please share your experience with the code by commenting or rating.

[1] Mathieu's Equation (Parametric Oscillator)

Link: (http://www.mathworks.com/matlabcentral/fileexchange/34381-mathieu-equation-parametric-oscillator)